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Exponentially fitted numerical method for solving singularly perturbed delay reaction-diffusion problem with nonlocal boundary condition

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Abstract

Objectives In this article, a singularly perturbed delay reaction-diffusion problem with nonlocal boundary conditions is considered. The exponential fitting factor is introduced to treat the solutions inside the boundary layer which occur due to perturbation parameter. The considered problem has interior layer at $s = 1$ and strong boundary layers at $s = 0$ and $s = 2$. We proposed an exponentially fitted finite difference method to solve the considered problem. The nonlocal boundary condition is treated using Composite Simpson's $\frac{1}{3}$ rule.

Result The stability and uniform convergence analysis of the proposed approach are established. The error estimation of the developed method is shown to be second-order uniform convergent. Two test examples were carried out to validate the applicability of the developed numerical method. The numerical results reflect the theoretical estimations.

Keywords Singularly perturbed problem, Reaction-diffusion problem, Nonlocal boundary condition

Mathematics Subject Classification Primary 65L11, 65L12, 65L20, 65L70

Introduction

Many problems in science can be described by differential equations involving small parameter and delay [1–3]. Such mathematical problems can be extremely difficult to solve exactly and, in such cases, approximate solutions are required. Various scientific and engineering processes can be modeled as integral terms over the spatial domain that appear inside or outside of the boundary conditions [4, 5]. Such problems are said to be nonlocal problems. Differential equations having nonlocal problems become singularly perturbed while the highest derivative is multiplied by a small parameter. Many physical phenomena are formulated as nonlocal mathematical models. For example, thermodynamics [6], plasma physics [7], heat conduction [8, 9], underground water flow and populace dynamics [10] can be decreased to the nonlocal problems with integration conditions. Singularly perturbed delay differential equations (SPDDEs) with nonlocal boundary conditions are observed to be an exciting and important type of problem, which plays a vital role in modelling a wide range of realistic phenomena and also broadly implemented in fields like bio-sciences, control-theory [11], HIV infection models [12], populace dynamics [13] and signal transition [14], and so forth.

The well posedness of singularly perturbed differential equations (SPDEs) with nonlocal boundary conditions

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was proved in [15, 16]. SPDEs with integral boundary conditions are an essential class of problems and are studied by several authors. The authors in [17] presented a numerical method based on FDM for solving a class of third order singularly perturbed convection diffusion type equations with integral boundary condition on a Shishkin mesh. Cimen and Cakir [18] construct an exponentially fitted difference scheme on an equidistant mesh for solving singularly perturbed nonlocal differential-difference problem with delay. The authors in [19] developed a numerical method based on FDM for solving a class of systems of singularly perturbed convection diffusion type equations with integral boundary conditions on a Shishkin mesh. Raja and Tamislevan [20] considered a class of system of singularly perturbed reaction diffusion equations with integral boundary conditions and developed a numerical method based on a finite difference scheme on a Shishkin mesh. In [21], the authors advanced a finite difference scheme on a suitable piecewise Shishkin type mesh for solving SPDDEs of convection-diffusion kind with integral boundary condition (IBC).

The authors in [22] investigated a class of third order SPDDEs of the convection-diffusion kind with IBC. They devised a numerical method depends on FDM with Shishkin mesh. Sekar and Tamilselvan [23] looked at a class of SPDDEs of convection-diffusion type with IBC. A FDM with suitable piecewise Shishkin type mesh was developed to solve the problem. The authors in [24] presented a numerical method depends on a FDM on Shishkin mesh to solve the third-order SPDDEs of reaction-diffusion kind with IBC. The authors in [25] used an exponentially fitted numerical scheme to solve SPDDEs of convection-diffusion kind with nonlocal boundary conditions. Debela and Duressa [26] improved the order of accuracy for the method proposed in [25]. Kumar and Kumari [27] developed the method based on the idea of B-spline functions and efficient numerical method on a piecewise-uniform mesh was recommended to approximate the solutions of SPDDEs with IBC.

The standard numerical schemes used for solving a class of SPDEs are sometime not well posed and fail to provide exact solution for very small perturbation parameter ε. Consequently, it is efficient to propose appropriate numerical schemes which are uniformly convergent. To the best of our knowledge, the singularly perturbed delay reaction-diffusion problem with nonlocal boundary conditions has not previously been numerically handled using an exponentially fitted numerical technique. The main aim of this work is to develop ε-uniform convergent numerical method for SPDDEs of the reaction-diffusion problem with nonlocal boundary conditions.

This article is organized in the following manner. In section "Introduction", a brief introduction of the given problem is discussed. In section "Properties of continuous problem", properties of continuous problem are given. In section "Formulation of numerical scheme", formulation of numerical scheme is given. Uniform convergence analysis is studied in section "Uniform convergence analysis". In section "Numerical examples and discussion", numerical examples and discussion are given. In section "Conclusion", conclusion of the article is given.

In this work, we use the following notations: $\Omega=(0,2)$, $\bar{\Omega}=[0,2]$, $\Omega_1=(0,1)$, $\Omega_2=(1,2)$, $\bar{\Omega}^{2N}=\{0,1,2,\cdots,2N\}$, $\Omega_1=\{1,2,3,\cdots,N-1\}$, $\Omega_2=\{N+1,N+2,\cdots,2N-1\}$. C denoted as a generic positive constant that are independent of parameter ε and $2N$ mesh points. We assume that $\sqrt{\varepsilon}\le CN^{-1}$. We used the maximum norm defined by $||w||_\Omega := \sup|w(s)|, s\in\Omega$ to study the convergence of the numerical solution.

Properties of continuous problem

Consider a class of SPDDEs with nonlocal boundary condition

$$\begin{cases} \mathcal{L}w(s) = -\varepsilon w''(s) + a(s)w(s) + b(s)w(s-1) = f(s), & s\in(0,2),\\ w(s)=\phi(s), & s\in[-1,0],\\ \mathcal{K}w(2) = w(2) - \varepsilon\int_0^2 g(s)w(s)ds = L. \end{cases} \tag{1}$$

where, ε is a small positive number ($0<\varepsilon\ll 1$). Assume that $a(s)\ge\alpha>0$, $b(s)\le\beta<0$, $\alpha+\beta>0$, $f(s)$, and $\phi(s)$ are sufficiently smooth functions and $g(s)$ is non negative monotone function and satisfy $\int_0^2 g(s)ds<1$. The above assumptions ensure that the problem (1) has a unique solution $w\in X=C^0(\Omega)\cap C^1(\Omega)\cap C^2(\Omega_1\cup\Omega_2)$. The problem (1) is equivalent to

$$\mathcal{L}w(s) = F(s), \tag{2}$$

with boundary conditions

$$\begin{cases} w(s)=\phi(s), s\in(-1,0),\\ w(1^-)=w(1^+), w'(1^-)=w'(1^+),\\ \mathcal{K}w(2) = w(2) - \varepsilon\int_0^2 g(s)w(s)ds = L. \end{cases} \tag{3}$$

where

$$\mathcal{L}w(s) = \begin{cases} \mathcal{L}_1 w(s) = -\varepsilon w''(s) + a(s)w(s), & s\in\Omega_1=(0,1),\\ \mathcal{L}_2 w(s) = -\varepsilon w''(s) + a(s)w(s) + b(s)w(s-1), & s\in\Omega_2=(1,2). \end{cases}$$

$$F(s) = \begin{cases} f(s) - b(s)\phi(s-1), & s\in\Omega_1,\\ f(s), & s\in\Omega_2. \end{cases}$$

Lemma 1 (Maximum principle): *Assume* $\theta(s)$ *be any function such that* $\theta(0) \geq 0$, $\mathcal{K}\theta(2) \geq 0$, $\mathcal{L}_1\theta(s) \geq 0$, $\forall s \in \Omega_1$, $\mathcal{L}_2\theta(s) \geq 0$, $\forall s \in \Omega_2$, *and* $\theta'(1+) - \theta'(1-) = [\theta'](1) \leq 0$, *then* $\theta(s) \geq 0$, $\forall s \in \bar{\Omega}$.

Proof We use proof by contradiction. Let us construct test function

$$r(s) = \begin{cases} \frac{1}{8} + \frac{s}{2}, & s \in [0,1], \\ \frac{3}{8} + \frac{s}{4}, & s \in [1,2]. \end{cases} \tag{4}$$

Note that $r(s) > 0, \forall s \in \bar{\Omega}$, $\mathcal{L}r(s) > 0$, $\forall s \in \Omega_1 \cup \Omega_2$, $r(0) > 0, \mathcal{K}r(2) > 0$ and $[r'](1) < 0$. Let $\mu = \max\left\{\frac{-\theta(s)}{r(s)}\right\}$. Then, there exists $s_0 \in \bar{\Omega}$ such that $\theta(s_0) + \mu r(s_0) = 0$ and $\theta(s) + \mu r(s) \geq 0, \forall s \in \bar{\Omega}$. Therefore, the function $(\theta + \mu r)(s)$ attains its minimum at $s = s_0$.

Suppose the lemma doesn't hold true, then $\mu > 0$.

Case (i): $s_0 = 0; 0 < (\theta + \mu r)(0) = \theta(0) + \mu r(0) = 0$.

Case (ii): $s_0 \in \Omega_1$, $0 < \mathcal{L}_1(\theta + \mu r)(s_0) = -\varepsilon(\theta + \mu r)''(s_0) + a(s_0)(\theta + \mu r)(s_0) \leq 0$. Case (iii): $s_0 = 1$, $0 \leq [\theta + \mu r]'(1) = [\theta'](1) + \mu r'(1) < 0$.

Case (iv): $s_0 \in \Omega_2, 0 < \mathcal{L}_2(\theta + \mu r)(s_0)$

$$\begin{aligned} &= -\varepsilon(\theta + \mu r)''(s_0) + a(s_0)(\theta + \mu r)(s_0) \\ &\quad + b(s_0)(\theta + \mu r)(s_0 - 1) \leq 0. \end{aligned}$$

Case (v): $s_0 = 2$, $0 \leq \mathcal{K}(\theta + \mu r)(2) = (\theta + \mu r) - \varepsilon \int_0^2 g(s)(\theta + \mu r)(s)ds \leq 0$.

Take note that in every case, we ended up with a contradiction. Hence $\mu > 0$ is impossible. Therefore $\theta(s) \geq 0$, $\forall s \in \bar{\Omega}$. □

Since the operator $\mathcal{L}$ satisfy the above maximum principle, the continuous solution $w(s)$ of the (2)-(3) is unique if it exists.

Lemma 2 (stability Result): *The solution* $w(s)$ *for the problems in* (1) *satisfies the bound*

$$|w(s)| \leq \max\{|w(0)|, |\mathcal{K}w(2)|, ||\mathcal{L}w||\}. \tag{5}$$

Proof This Lemma can be proved using Lemma 1 and by constructing a barrier functions as $\psi^{\pm}(s) = \max\{|w(0)|, |\mathcal{K}w(2)|, ||\mathcal{L}w||\}r(s) \pm w(s),\ s \in \bar{\Omega}$, where $r(s)$ is a test functions in (4). □

Lemma 3 *Let* $w \in C^2(\Omega)$ *be the solution of* (1). *Then, for* $k = 1, 2, 3, 4$,

$$\left|\left|w^{(k)}\right|\right| \leq C\left(1 + \varepsilon^{-k/2}\right). \tag{6}$$

Proof Using Lemma 2 and applying arguments as given in [28] this lemma gets proved. □

Formulation of numerical scheme

The problems in (1) manifest strong boundary layers at $s = 0$ and $s = 2$ and have interior layer at $s = 1$. Due to a dependence of $a(s)$ and $b(s)$ on spatial variable s, we cannot solve the problem analytically. With N identical mesh points, the range [0, 2] is separated into $\Omega_1 = (0, 1)$ and $\Omega_2 = (1, 2)$. After that, we get $s_i = ih, i = 0, 1, 2, \cdots, 2N$. The differential equation is obtained if we take into account the interval $s \in (0, 1)$ and the coefficients of (1) are assessed on the midpoint of each interval.

$$\begin{cases} -\varepsilon w''(s) + a(s)w(s) = f(s) - b(s)\phi(s-1), & s \in \Omega_1 = (0,1), \\ w(0) = \phi(0), \\ w(1) = \gamma, \end{cases} \tag{7}$$

where γ is any arbitrary constant. Now, we present exponentially fitted operator finite difference method (FOFDM) on the discretized domain $\Omega_1 = [0, 1]$. From (7) we have

$$-\varepsilon w''(s) + a(s)w(s) = F(s), \quad s \in \Omega_1 = (0,1), \tag{8}$$

where $F(s) = f(s) - b(s)\phi(s-1)$.

We employ the theory used in the asymptotic technique to solve a singularly perturbed BVPs to find an approximation to the solution of (8). In our scenario, the domain is separated into three sub-domains, two boundary-layer subdomains near $s = 0$ and $s = 1$ and one regular subdomain, and the boundary layer problem is changed to a regular problem by proper transformations using stretching variables. We looked at the asymptotic expansion solution to the problem in (8) based on the theory of singular perturbations presented in [29].

$$w(s, \varepsilon) = \sum_{i=0}^{N} [w_i(s) + v_i(\tau) + u_i(\eta)]\varepsilon^i, \tag{9}$$

where $\tau = \frac{s}{\sqrt{\varepsilon}}$, $\eta = \frac{1-s}{\sqrt{\varepsilon}}$. Then, the zeroth order of (9) asymptotic expansion is given as

$$w(s) = w_0(s) + v_0(\tau) + u_0(\eta),$$

where $w_0(s) = \frac{F(s)}{a(s)}$ is a solution of a reduced problem (1), which does not satisfy the boundary conditions, $v_0 = Ae^{-\sqrt{\frac{a(0)}{\varepsilon}}s}$ is the left boundary layer correction and $w_0 = Be^{-\sqrt{\frac{a(1)}{\varepsilon}}(1-s)}$ is the right boundary layer correction. Therefore, the asymptotic solution of the zeros order of (7) become

$$w(s) = w_0(s) + Ae^{-\sqrt{\frac{a(0)}{\varepsilon}}s} + Be^{-\sqrt{\frac{a(1)}{\varepsilon}}(1-s)} + \mathcal{O}(\varepsilon), \tag{10}$$

where A and B are determined using the given boundary conditions. Now, we separate the range [0, 1] into N equal parts with uniform mesh length h. Let $0 = s_0, s_1, \cdots, s_N = 1$ be the mesh points. Then, we have $s_i = ih$; $i = 0, 1, \cdots N$. We choose N_1 and N_2 such that $s_{N_1} = \sqrt{\varepsilon}$ and $s_{N_2} = 1 - \sqrt{\varepsilon}$. Then, in the range $[0, \sqrt{\varepsilon}\,]$ the boundary layer at $s = 0$ and in the range $[1 - \sqrt{\varepsilon}, 1]$, the boundary layer will be at $s = 1$.

At $s = s_i$, the above differential equations (7) can be written as

$$\begin{cases} -\varepsilon w''(s_i) + a(s_i)w(s_i) = F(s_i), \\ w(0) = \phi(0) \\ w(N) = \gamma. \end{cases} \tag{11}$$

For convenience, we take $a(s_i) = a_i, w(s_i) = w_i, F(s_i) = F_i$. Now, consider finite difference for $w_i'' = \dfrac{w_{i-1} - 2w_i + w_{i+1}}{h^2}$, $i = 1, 2, 3, \cdots, N-1$ and by substituting in (11), we obtain

$$-\varepsilon\left(\frac{w_{i-1} - 2w_i + w_{i+1}}{h^2}\right) + a_i w_i = F_i. \tag{12}$$

Case I: left boundary layer

The problem of the form in (7) has left boundary layer at interval $[0, \sqrt{\varepsilon}]$. Then, the zeroth order approximation of asymptotic solution is given as

$$w(s) = w_0(s) + Ae^{\sqrt{\frac{a(0)}{\varepsilon}}s} + \mathcal{O}(\varepsilon), \tag{13}$$

where $w_0(s)$ is the solution of the reduced problem and we choose A as a suitable constant. Using Taylor series approximation for $w_0(i+1)h$ and $w_0(i-1)h$ up to first order, we obtain

$$w(s_{i+1}) = w_0(ih) + Ae^{-\sqrt{a(0)}(i+1)\rho}, \tag{14}$$

$$w(s_{i-1}) = w_0(ih) + Ae^{-\sqrt{a(0)}(i-1)\rho}, \tag{15}$$

where $\rho = h/\sqrt{\varepsilon}$ and $h = 1/N$. To handle the oscillation of the perturbation parameter, we multiply exponentially fitting factor σ_1 for the term with a perturbation parameter as,

$$-\varepsilon\sigma_1 w''(s) + a(s)w(s) = F(s), \tag{16}$$

with boundary conditions $w(0) = \phi(0)$ and $w(1) = \gamma$.

If W_i is a discrete solution for $w(s)$ at grid point s_i, the numerical method for (16) is written in operator form as

$$\mathcal{L}^h W_i = F_i, \quad i = 1, 2, 3, \cdots, N-1.$$

with boundary conditions $W(0) = \phi(0)$, $W(N) = \gamma$, where

$$\mathcal{L}^h W_i = -\varepsilon\sigma_1\left(\frac{W_{i-1} - 2W_i + W_{i+1}}{h^2}\right) + a_i W_i = F_i. \tag{17}$$

From (17), we have

$$-\frac{\sigma_1}{\rho^2}(W_{i-1} - 2W_i + W_{i+1}) = F_i - a_i W_i.$$

Now, by taking the limit as $h \to 0$ and using (13)–(15) and manipulate some calculations, the exponential fitting factor is obtained as

$$\sigma_1 = \frac{\rho^2 a(0)}{4}\left(\csc h\left(\frac{\rho}{2}\sqrt{a(0)}\right)\right)^2. \tag{18}$$

This will be a fitting factor in the left boundary layer.

Case II: right boundary layer

In the interval $[1 - \sqrt{\varepsilon}, 1]$, the right boundary layer will be on the right side near to $s = 1$. Now we introduce the exponential fitting factor as

$$-\varepsilon\sigma_2\left(\frac{W_{i-1} - 2W_i + W_{i+1}}{h^2}\right) + a_i W_i = F_i, \ \ i = 1, 2, 3, \cdots, N-1. \tag{19}$$

with boundary condition $W(0) = \phi(0)$ and $W(N) = \gamma$.

Now, to introduce the fitting factor σ_2 on the right hand side, we use the right boundary layer asymptotic solution with outer layer

$$w_i = w_0(s_i) + Be^{-\sqrt{\frac{a(1)}{\varepsilon}}(1-s_i)}, \tag{20}$$

where $w_0(s_i)$ is the solution of the reduced problem and B is arbitrary constant determined by using boundary condition. Using the same fashion as the left boundary layer case, the exponentially fitting factor is obtained as

$$\sigma_2 = \frac{\rho^2 a(0)}{4}\left(\csc h\left(\frac{\rho}{2}\sqrt{a(1)}\right)\right)^2. \tag{21}$$

The required discrete problem become given as

$$\mathcal{L}^h W_i = -\varepsilon\sigma_2\left(\frac{W_{i-1} - 2W_i + W_{i+1}}{h^2}\right) + a_i W_i = F_i, \quad i = 1, 2, 3, \cdots, N-1$$

with boundary conditions $W(0) = \phi(0)$ and $W(N) = \gamma$.

An exponential fitting factor over $\Omega_2 = (1, 2)$ is analogously calculated as a fitting factor in $\Omega_1 = (1, 2)$. In general, one can take an artificial viscosity (fitting factor) for the given problem on $\bar{\Omega}_i^{2N}$ as

$$\sigma(\rho) = \frac{\rho^2 a(0)}{4}\left(\csc h\left(\frac{\rho}{2}\sqrt{\alpha}\right)\right)^2.$$

Suppose that $\bar{\Omega}^{2N}$ denote a separation of $[0, 2]$ into $2N$ subintervals such that $0 = s_0 < s_1 < s_2 < s_3 < \cdots < s_N = 1$, and $1 < s_{N+1} < s_{N+1} < \cdots < s_{2N} = 2$ with $h_i = s_i - s_{i-1}, h = 2/2N = 1/N, \ i = 1, 2, \cdots, 2N$.

Case I: Consider equation (1) on the domain $\Omega_1 = (0, 1)$ which is given by

$$-\varepsilon\sigma(\rho)w''(s) + a(s)w(s) = f(s) - \phi(s-1).$$

Hence, the required difference equation becomes

$$-\frac{\varepsilon\sigma}{h^2}W_{i-1} + \left(\frac{2\varepsilon\sigma}{h^2} + a_i\right) W_i - \frac{\varepsilon\sigma}{h^2}W_{i+1} = f_i - b_i\phi(s_i - N), \tag{22}$$

for $i = 1, 2, 3, \cdots, N$. Equation (22) can be rewritten as

$$A_i W_{i-1} + B_i W_i + C_i W_{i+1} = H_i,$$

where $A_i = -\frac{\varepsilon\sigma}{h^2}, \ B_i = \frac{2\varepsilon\sigma}{h^2} + a_i, \ C_i = -\frac{\varepsilon\sigma}{h^2}, \ H_i = f_i - b_i\phi(s_i - N)$.

Case II: Consider equation (1) on the domain Ω_2. Then, equation (1) also has left boundary layer near $s = 1$ and right boundary layer at $s = 2$. Then, by applying exponentially fitted finite difference scheme, we obtain $-\varepsilon\sigma(\rho)\left(\frac{w_{i-1}-2w_i+w_{i+1}}{h^2}\right) + a_i w_i + b_i w_{i-N} + \tau = f_i$, which is rewritten as

$$E_i W_{i-1} + F_i W_i + G_i W_{i+1} + b_i W_{i-N} = f_i \tag{23}$$

where $E_i = -\frac{\varepsilon\sigma}{h^2}, \ F_i = \frac{2\varepsilon\sigma}{h^2} + a_i, \ G_i = -\frac{\varepsilon\sigma}{h^2}$.

Case III: For $i = 2N$, we approximate $\int_0^2 g(s)w(s)ds$ using the composite Simpson's $\frac{1}{3}$ rule.

$$\int_0^2 g(s)w(s)ds = \frac{h}{3}\left(g(0)w(0) + g(2)w(2) + 2\sum_{i=1}^{2N-1} g(s_{2i})w(s_{2i}) + 4\sum_{i=1}^{2N} g(s_{2i-1})w(s_{2i-1})\right) = L. \tag{24}$$

Since, $w(0) = \phi(0)$, from (3), this equation can be rewritten as

$$-\frac{4\varepsilon h}{3}\sum_{i=1}^{2N} g(s_{2i-1})W(s_{2i-1}) - \frac{2\varepsilon h}{3}\sum_{i=1}^{2N-1} g(s_{2i})W(s_{2i}) + \left(1 - \frac{\varepsilon h g(2)}{3}\right)W(s_{2N}) = L$$

As a result, the fundamental schemes for solving (1) on the entire domain $\bar{\Omega} = [0, 2]$ are the schemes given in (22)–(23) and (24), together with the local truncation error of τ.

Uniform convergence analysis

The discrete solution corresponding to equation (1) are given as follows:

$$\mathcal{L}_1 W_i = -\varepsilon D^+ D^- W_i + a_i W_i = f_i - b_i\phi_{i-N}, \quad i = 1, 2, 3, \cdots N-1, \tag{25}$$

$$\mathcal{L}_2 W_i = -\varepsilon D^+ D^- W_i + a_i W_i + b_i W_{i-N} = f_i, \quad i = N+1, N+2, N+3, \cdots 2N-1, \tag{26}$$

subject to the boundary conditions:

$$\begin{cases} W_i = \phi_i, \quad i = -N, -N+1, \cdots, 0 \\ \mathcal{K}^N W_{2N} = W_{2N} - \sum_{i=1}^{2N} \frac{g_{i-1}W_{i-1} + g_i W_i + g_{i+1}W_{i+1}}{3} h_i \quad \text{and} \\ D^- W_N = D^+ W_N. \end{cases}$$

Lemma 4 (Discrete Maximum principle): *Assume*

$$\sum_{i=1}^{2N} \frac{g_{i-1} + g_i + g_{i+1}}{3} h_i = \lambda < 1$$

and $\theta(s_i)$ *be any function such that* $\theta(s_0) \geq 0, \mathcal{K}\theta(s_{2N}) \geq 0$, $\mathcal{L}_1\theta(s_i) \geq 0,\ \forall s_i \in \Omega_1^{2N}$, $\mathcal{L}_2\theta(s_i) \geq 0,\ \forall s_i \in \Omega_2^{2N}$, *and* $D^+(\theta(s_N)) - D^-(\theta(s_N)) \leq 0$, *then* $\theta(s_i) \geq 0, \forall s_i \in \bar{\Omega}^{2N}$.

Proof Define the test function

$$r(s_i) = \begin{cases} \frac{1}{8} + \frac{s_i}{2}, & s_i \in [0,1] \cap \Omega^{2N}, \\ \frac{3}{8} + \frac{s_i}{4}, & s_i \in [1,2] \cap \Omega^{2N}, \end{cases} \tag{27}$$

Note that $r(s_i) > 0, \forall s_i \in \bar{\Omega}^{2N}$, $\mathcal{L}r(s_i) > 0$, $\forall s_i \in \Omega_1^{2N} \cup \Omega_2^{2N}$, $r(s_0) > 0$, $\mathcal{K}r(s_{2N}) > 0$ and $[r'](N) < 0$. Let $\mu = \max\left\{\frac{-\theta(s_i)}{r(s_i)}; s_i \in \bar{\Omega}^{2N}\right\}$. Then, there exists $s_0 \in \bar{\Omega}$ such that $\theta(s_0) + \mu r(s_0) = 0$ and $\theta(s_i) + \mu r(s_i) \geq 0, \forall s_i \in \bar{\Omega}$. Therefore, the function $(\theta + \mu r)(s_i)$ attains its minimum at $s = s_k$.

Suppose the lemma doesn't hold true, then $\mu > 0$.

Case (i): $s_k = 0; 0 < (\theta + \mu r)(0) = \theta(0) + \mu r(0) = 0$.

Case (ii): $s_k \in \Omega_1^{2N}$, $0 < \mathcal{L}_1(\theta + \mu r)(s_k) = -\varepsilon(\theta + \mu r)''(s_k) + a(s_k)(\theta + \mu r)(s_k) \leq 0$.

Case (iii): $s_k = s_N$, $0 \leq D[\theta + \mu r]'(s_N) = [\theta'](s_N) + \mu r'(s_N) < 0$.

Case (iv): $s_k \in \Omega_2, 0 < \mathcal{L}_2(\theta + \mu r)(s_k) =$

$$\begin{aligned} &-\varepsilon(\theta + \mu r)''(s_k) + a(s_k)(\theta + \mu r)(s_k) \\ &+ b(s_k)(\theta + \mu r)(s_k - N) \leq 0. \end{aligned}$$

Case (v): $s_k = s_{2N},\ 0 \leq \mathcal{K}(\theta + \mu r)(s_{2N}) = (\theta + \mu r)s_{2N}$

$$-\sum_{i=1}^{2N} \frac{g_{i-1}(\theta + \mu r)s_{i-1} + g_i(\theta + \mu r)s_i + g_{i+1}(\theta + \mu r)s_{i+1}}{3} h_i \leq 0.$$

Take note that in every cases, we arrive at a contradiction. Therefore $\mu > 0$ is impossible. Hence, $\theta(s_i) \geq 0$, $\forall s_i \in \bar{\Omega}^{2N}$. □

Since the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ satisfy the above maximum principle, the discrete solution W_i of the (25)-(26) is unique if it exists.

Lemma 5 *Let* $\psi(s_i)$ *be any mesh function. Then for* $0 \leq i \leq 2N$ *we have the following estimate.*

$$|\psi(s_i)| \leq \max\left\{|\psi(s_0)|, |\mathcal{K}\psi(s_{2N})|, \max_{i \in \Omega_1 \cup \Omega_2} \left|\mathcal{L}^{2N}\psi(s_i)\right|\right\} \tag{28}$$

Proof The proof is follows from Lemma 4 and by constructing a barrier functions

$$\begin{aligned} \theta^{\pm}(s_i) &= \max\left\{|\psi(s_0)|, |\mathcal{K}\psi(s_{2N})|, \max_{i \in \Omega_1 \cup \Omega_2} \left|\mathcal{L}^{2N}\psi(s_i)\right|\right\} \\ &\quad r_i \pm \psi(s_i), \quad \forall s_i \in \bar{\Omega}^{2N}. \end{aligned}$$

r_i is a test function given in (27). □

Theorem 1 *Let* $w(s_i)$ *and* $W(s_i)$ *be the continuous solution of* (1) *and discrete solutions of* (22)-(24) *respectively. Then, for sufficiently large N, the following truncation error estimate holds*:

$$\sup_{1 \leq i \leq 2N} |W(s_i) - w(s_i)| \leq CN^{-2} \tag{29}$$

Proof Let us define a local truncation error as

$$\begin{aligned} \left|\mathcal{L}^{2N}(W(s_i) - w(s_i))\right| &= \left|\mathcal{L}^{2N}W(s_i) - \mathcal{L}^{2N}w(s_i)\right| \\ &= \left|-\varepsilon\sigma D^+D^-W(s_i) - \left(-\varepsilon\frac{d^2}{dx^2}w(s_i)\right)\right| \\ &\leq \left|\varepsilon\frac{d^2}{ds^2}w(s_i) - \varepsilon\sigma D^+D^-W(s_i)\right| \end{aligned}$$

where $\sigma(\rho) = \frac{\rho^2 a(0)}{4}\csc h^2\left(\frac{\rho}{2}\sqrt{\alpha}\right)$ and $\rho = N^{-1}/\sqrt{\varepsilon}$. From Taylor series expansion we get the bounds as

Table 1 Maximum absolute error and rate of convergence of the scheme for Example 1

$\varepsilon \downarrow N \mapsto$	2^8	2^9	2^{10}	2^{11}	2^{12}
2^{00}	1.5590e-07	3.8974e-08	9.7458e-09	2.4338e-09	5.7258e-10
	2.0000	1.9997	2.0016	2.0877	
2^{-02}	5.8936e-07	1.4735e-07	3.6837e-08	9.2184e-09	2.2931e-09
	1.9999	2.0000	1.9986	2.0072	
2^{-04}	2.0742e-06	5.1865e-07	1.2967e-07	3.2418e-08	8.1199e-09
	1.9997	1.9999	2.0000	1.9973	
2^{-06}	8.2661e-06	2.0681e-06	5.1714e-07	1.2929e-07	3.2323e-08
	1.9989	1.9997	1.9999	2.0000	
2^{-08}	3.4244e-05	8.5876e-06	2.1486e-06	5.3725e-07	1.3432e-07
	1.9955	1.9989	1.9997	1.9999	
2^{-10}	1.3923e-04	3.5237e-05	8.8363e-06	2.2108e-06	5.5280e-07
	1.9823	1.9956	1.9989	1.9997	
2^{-12}	5.3954e-04	1.4155e-04	3.5824e-05	8.9834e-06	2.2476e-06
	1.9304	1.9823	1.9956	1.9989	
2^{-14}	1.8128e-03	5.4440e-04	1.4281e-04	3.6140e-05	9.0627e-06
	1.7355	1.9306	1.9824	1.9956	
2^{-16}	3.9145e-03	1.8217e-03	5.4691e-04	1.4346e-04	3.6304e-05
	1.1035	1.7359	1.9307	1.9824	
E^N	3.9145e-03	1.8217e-03	5.4691e-04	1.4346e-04	3.6304e-05
R^N	1.1035	1.7359	1.9307	1.9824	

$$
\begin{aligned}
\left|D^+D^-W(s_i)\right| &\le C\left|\frac{d^2W(s_i)}{ds^2}\right|,\\
\left|\left(\frac{d^2}{ds^2}-D^+D^-\right)W(s_i)\right| &\le CN^{-2}\left|\frac{d^4W(s_i)}{ds^4}\right|.
\end{aligned}
\tag{30}
$$

Using the bounds for the differences of the derivatives in (30) and based on the result given in [30], we have

$$
\begin{aligned}
\left|\mathcal{L}^{2N}(W(s_i)-w(s_i))\right| \le& C\left(|\varepsilon\sigma-\varepsilon|\left|W''(s_i)\right| + \varepsilon h^2\left|W^{(4)}(s_i)\right|\right)\\
\le& \varepsilon CN^{-2}\left|\frac{d^4W(s_i)}{ds^4}\right|
\end{aligned}
$$

Here, the target is to show the scheme is convergent independent of the number of mesh points. By using the bounds for the derivatives of the solution in Lemma (3), we obtain

$$
\begin{aligned}
\mathcal{L}^{2N}(W(s_i)-w(s_i)) \le& \varepsilon CN^{-2}\left(1+\varepsilon^{-2}\right)\\
\le& \varepsilon CN^{-2}+\varepsilon^{-1}CN^{-2}\\
\le& CN^{-2}, \text{ since } \varepsilon^{-1}>\varepsilon.
\end{aligned}
$$

Hence by discrete maximum principle, we obtain

$$
|W(s_i)-w(s_i)| \le CN^{-2}. \tag{31}
$$

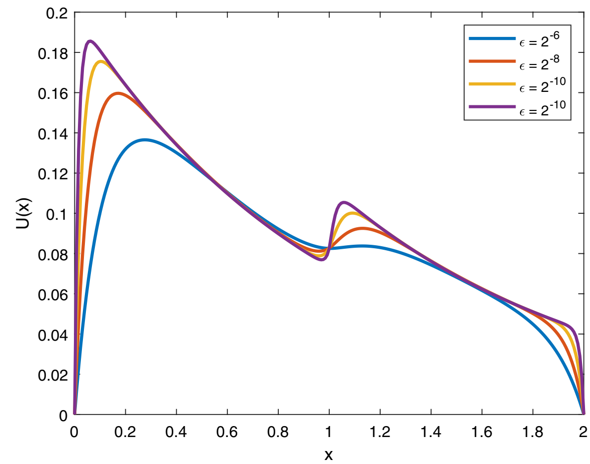

Fig. 1 Graph of numerical solution which displays the existing layer for Example 1

At the point $s_i = s_{2N}$, we have

$$
\begin{aligned}
&\mathcal{K}^{2N}(W(s_i)-w(s_i))\\
=&\mathcal{K}^{2N}W(s_{2N})-\mathcal{K}^{2N}w(s_i),\\
=&\phi_r-\mathcal{K}^{2N}W(s_{2N}),\\
=&\mathcal{K}w(s_i)-\mathcal{K}^{2N}W(s_{2N}),\\
=&w(s_{2N})-\varepsilon\int_0^2 g(s)w(s)ds-\left(w(s_{2N})-\varepsilon\int_{s_0}^{s_{2N}} g(s)w(s)ds\right),\\
=&\varepsilon\int_{s_0}^{s_{2N}} g(s)w(s)ds-\varepsilon\sum_{i=1}^{2N}\frac{g_{i-1}w_{i-1}+4g_iw_i+g_{i+1}w_{i+1}}{3}h\\
\le& C\varepsilon h^4\left(w^{(4)}(\xi_1)+w^{(4)}(\xi_2)+\cdots+w^{(4)}(\xi_{2N})\right)\\
\le& C\varepsilon h^4\left\|\frac{d^4w(\xi_i)}{dx^4}\right\|\\
\le& C\varepsilon h^4(1+\varepsilon^{-2})\\
\le& C\varepsilon h^4+Ch^4\varepsilon^{-1}\\
\le& Ch^2\\
=&CN^{-2}.
\end{aligned}
$$

By using the bounds for derivative of the solution in Lemma 3 and applying discrete maximum principle, we obtain

$$
||W(s_i)-w(s_i)|| \le CN^{-2}. \tag{32}
$$

Thus, the results of (31) and (32) shows (29). Hence the proof is complete. □

Table 2 Maximum absolute error and rate of convergence of the scheme for Example 2

$\varepsilon \downarrow N \mapsto$	2^8	2^9	2^{10}	2^{11}	2^{12}
2^{00}	1.0861e-07	2.7323e-08	6.8515e-09	1.7155e-09	4.2928e-10
	1.9910	1.9956	1.9978	1.9986	
2^{-02}	6.6233e-07	1.6559e-07	4.1397e-08	1.0372e-08	2.5643e-09
	1.9999	2.0000	1.9968	2.0161	
2^{-04}	2.5661e-06	6.4162e-07	1.6041e-07	4.0103e-08	1.0065e-08
	1.9998	2.0000	2.0000	1.9944	
2^{-06}	9.6604e-06	2.4168e-06	6.0430e-07	1.5108e-07	3.7770e-08
	1.9990	1.9998	2.0000	2.0000	
2^{-08}	3.7499e-05	9.4020e-06	2.3522e-06	5.8816e-07	1.4705e-07
	1.9958	1.9990	1.9997	1.9999	
2^{-10}	1.4619e-04	3.6986e-05	9.2740e-06	2.3202e-06	5.8017e-07
	1.9828	1.9957	1.9989	1.9997	
2^{-12}	5.5373e-04	1.4516e-04	3.6729e-05	9.2100e-06	2.3042e-06
	1.9315	1.9827	1.9956	1.9989	
2^{-14}	1.8396e-03	5.5161e-04	1.4464e-04	3.6601e-05	9.1780e-06
	1.7377	1.9312	1.9825	1.9956	
2^{-16}	3.9534e-03	1.8351e-03	5.5055e-04	1.4439e-04	3.6537e-05
	1.1072	1.7369	1.9309	1.9825	
E^N	3.9534e-03	1.8351e-03	5.5055e-04	1.4439e-04	3.6537e-05
R^N	1.1072	1.7369	1.9309	1.9825	

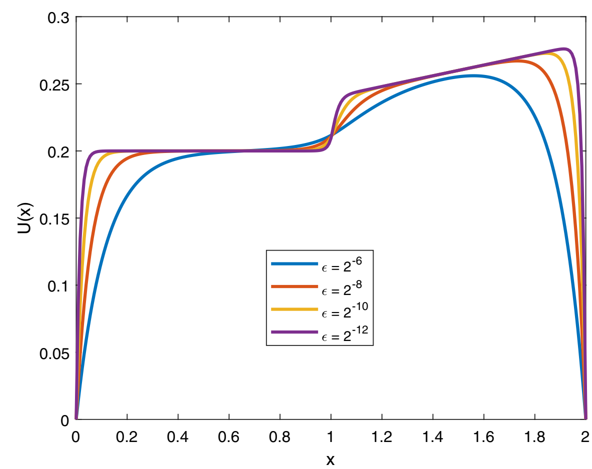

Fig. 2 Graph of numerical solution which displays the existing layer for Example 2

Numerical examples and discussion

Since the exact solution of the given examples is not available, we use a double mesh technique to compute the maximum pointwise absolute error of the presented method.

Example 1

$$-\varepsilon\frac{d^2w(s)}{ds^2}+5w(s)-w(s-1)=e^{-s}$$
$$w(s)=1,\ \ s\in[-1,0]$$
$$\mathcal{K}w(2)=w(2)-\varepsilon\int_0^2\frac{s}{3}w(s)ds=0.$$

Example 2

$$-\varepsilon\frac{d^2w(s)}{ds^2}+5w(s)-sw(s-1)=1$$
$$w(s)=1,\ \ s\in[-1,0]$$
$$\mathcal{K}w(2)=w(2)-\varepsilon\int_0^2\frac{1}{6}w(s)ds=0.$$

We define the maximum pointwise absolute error as $E_\varepsilon^N=\max_i\left|W_i^N-W_i^{2N}\right|$. where N is a number of mesh points. Next, we compute the ε-uniform error estimate by using the formula $E^N=\max_\varepsilon\left(E^N\right)$. We compute the rate of convergence of the method by using the formula $R_\varepsilon^N=\log_2\left(\frac{E_\varepsilon^N}{E_\varepsilon^{2N}}\right)$. In the same manner we compute the ε-uniform rate of convergence by using the formula $R^N=\log_2\left(\frac{E^N}{E^{2N}}\right)$. The assumption $\sqrt{\varepsilon}\leq CN^{-1}$ is made only for theoretical purpose. The numerical method works for all ε for our examples.

The solutions of the given examples manifest strong boundary layer of thickness $\mathcal{O}(\sqrt{\varepsilon})$ close to $s=0$ and $s=2$ and interior layer at $s=1$. Tables 1 and 2 indicates the maximum absolute error and rate of convergence of the scheme for Example 1 and 2 respectively. The given tables suggested that the developed scheme is a parameter uniform convergent independent of mesh points with second-order of convergence. Figures 1 and 2 shows a graph of a numerical solution which displays the formation of boundary layer and interior layer as ε goes to zero for Example 1 and 2 respectively.

Conclusion

A class of singularly perturbed delay differential equations of reaction-diffusion problem with nonlocal boundary conditions is solved numerically. Due to the presence of a perturbation parameter on the higher order derivative the solution of the problem exhibit a boundary layers at $s=0$ and $s=2$ and interior layer at $s=1$. To obtain a numerical solution for this types of problems, we developed an exponentially fitted operator finite difference method (FOFDM) on a uniform mesh. The nonlocal boundary condition is approximated using Simpson's $\frac{1}{3}$ rule. The stability and uniform convergence of the presented method are also investigated. Two test examples are considered for numerical experimentation to validate the applicability of the method. The developed numerical method is proved to be second-order uniformly convergent independent of perturbation parameter. Numerical experiments corroborate the theoretical findings. In future work, one can use a layer adapted mesh to solve the problem numerically.

Limitations

The developed numerical scheme is not layer resolving method (i.e. there is no sufficient number of mesh points in the boundary layer region).

Abbreviations

SPDEs	Singularly perturbed differential equations
SPDDEs	Singularly perturbed delay differential equations
IBC	Integral boundary conditions
FDM	Finite difference method

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Author contributions

GMW and MMW designed the method and write the draft. GFD and TGD reviewed and edited the draft. All authors read and approved the final manuscript.

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Declarations

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Not applicable.

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