

RESEARCH NOTE

Open Access



Generalized fixed point results of rational type contractions in partially ordered metric spaces

N. Seshagiri Rao^{1*} and K. Kalyani²

Abstract

Objectives: We investigated the existence and uniqueness of a fixed point for the mapping satisfying generalized rational type contraction conditions in metric space endowed with partial order. Suitable examples are presented to justify the results obtained.

Result: Some new fixed point results have been obtained for a mapping fulfilling generalized contractions. The uniqueness of the fixed point is also the part of the study based on an ordered relation. One example is given for a result which is not valid in the usual metric space.

Keywords: Partially ordered metric spaces, Generalized rational contractions, Fixed point, Ordered relation, Integral contractions

Mathematics Subject Classification: 54H25, 47H10

Introduction

First the idea of fixed point theory was introduced by H. Poincaré in 1886. Subsequently M. Fréchet in 1906 has given the fixed point theorem in terms of taking distance between the points and also the corresponding images of the operator at those points in metric spaces. Later in 1922, Banach has proven a fixed theorem for a contraction mapping in complete metric space. This principle plays a crucial role in several branches of mathematics. It is an important tool for finding the solutions of many existing results in nonlinear analysis. Besides, this renowned classical theorem offers an iteration method through that we are able to acquire higher approximation to the fixed point. This result has rendered a key role in finding systems of linear algebraic equations involving

iteration method. Iteration procedures are using in every branch of applied mathematics, convergence proof and also in estimating the process of errors, very often by an application of Banach's fixed point theorem.

Since then several authors have generalized this classical Banach's contraction theorem in an usual metric space and extensively reported in their work by taking various contraction conditions on the mappings, the readers may refer to [1–12]. Moreover, various generalizations of this result have been obtained by weakening its hypothesis in numerous spaces like rectangular metric spaces, pseudo metric spaces, fuzzy metric spaces, quasi metric spaces, quasi semi-metric areas, probabilistic metric spaces, D -metric spaces, G -metric spaces, F -metric spaces, cone metric spaces, some of which can be found in [13–28]. A lot of work on the results of fixed points, common fixed points, coupled fixed points in partly ordered metric spaces with different topological properties involved can be found from [29–41]. Some generalized fixed points results of monotone mappings in partially ordered b -metric spaces have been investigated

*Correspondence: seshu.namana@gmail.com

¹ Department of Applied Mathematics, School of Applied Natural Sciences, Adama Science and Technology University, Post Box No. 1888, Adama, Ethiopia

Full list of author information is available at the end of the article



© The Author(s) 2021. **Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>. The Creative Commons Public Domain Dedication waiver (<http://creativecommons.org/publicdomain/zero/1.0/>) applies to the data made available in this article, unless otherwise stated in a credit line to the data.

by Seshagiri Rao et al. [43, 46, 47], Kalyani et al. [42, 45, 48] and Belay Mitiku et al. [44]. Acar [49] explored some fixed point results of F -contraction for multivalued integral type mapping on a complete metric space. Recently, the notation of Ćirić type rational graphic (Y, Λ) -contraction pair mappings have been used and produced some new common fixed point results on partial b -metric spaces endowed with a directed graph G by Eskandar et al. [50].

The aim of this paper is to prove some fixed point results of a mapping in the frame work of a metric space endowed with partial order satisfying generalized contractive conditions of rational kind. The uniqueness of a fixed point is discussed through an ordered relation in a partially ordered metric space. Also, the conferred results generalize and extend a few well-known results of [20, 26] in the literature. Appropriate examples are highlighted to support the prevailing results.

Preliminaries

We start this section with the following subsequent definitions which are used frequently in our study.

Definition 1 [36] The triple $(\mathcal{Q}, \varrho, \leq)$ is called partially ordered metric spaces if (\mathcal{Q}, \leq) could be a partial ordered set and (\mathcal{Q}, ϱ) be a metric space.

Definition 2 [36] If ϱ is complete metric, then $(\mathcal{Q}, \varrho, \leq)$ is called complete partially ordered metric space.

Definition 3 [36] A partially ordered metric space $(\mathcal{Q}, \varrho, \leq)$ is called an ordered complete (OC), if for every convergent sequence $\{\bar{U}_n\} \subset \mathcal{Q}$, the subsequent condition holds: either

- if a non-increasing sequence $\bar{U}_n \rightarrow \bar{U} \in \mathcal{Q}$, then $\bar{U} \leq \bar{U}_n$, for all $n \in \mathbb{N}$, that is, $\bar{U} = \inf\{\bar{U}_n\}$, or
- if $\bar{U}_n \in \mathcal{Q}$ is a non-decreasing sequence such that $\bar{U}_n \rightarrow \bar{U}$ implies that $\bar{U}_n \leq \bar{U}$, for all $n \in \mathbb{N}$, that is, $\bar{U} = \sup\{\bar{U}_n\}$.

Definition 4 [36] A map $\mathcal{J} : \mathcal{Q} \rightarrow \mathcal{Q}$ is a non-decreasing, if for every $\bar{U}, \varepsilon \in \mathcal{Q}$ with $\bar{U} < \varepsilon$ implies that $\mathcal{J}\bar{U} \geq \mathcal{J}\varepsilon$.

Main text

We begin this section with the subsequent result.

Theorem 1 Let $(\mathcal{Q}, \varrho, \leq)$ be a complete partially ordered metric space. Suppose a self-map \mathcal{J} on \mathcal{Q} is

continuous, non-decreasing and satisfies the contraction condition

$$\begin{aligned} & \varrho(\mathcal{J}\bar{U}, \mathcal{J}\varepsilon) \\ & \leq \alpha \frac{\varrho(\bar{U}, \mathcal{J}\bar{U})\varrho(\varepsilon, \mathcal{J}\varepsilon)}{\varrho(\bar{U}, \varepsilon)} + \beta[\varrho(\bar{U}, \mathcal{J}\bar{U}) + \varrho(\varepsilon, \mathcal{J}\varepsilon)] \\ & + \gamma\varrho(\bar{U}, \varepsilon) + \mathcal{L} \min\{\varrho(\bar{U}, \mathcal{J}\varepsilon), \varrho(\varepsilon, \mathcal{J}\bar{U})\}, \end{aligned} \quad (1)$$

for any $\bar{U} \neq \varepsilon \in \mathcal{Q}$ with $\bar{U} \leq \varepsilon$, where $\mathcal{L} \geq 0$, and $\alpha, \beta, \gamma \in [0, 1)$ with $0 \leq \alpha + 2\beta + \gamma < 1$. If $\bar{U}_0 \leq \mathcal{J}\bar{U}_0$ for certain $\bar{U}_0 \in \mathcal{Q}$, then \mathcal{J} has a fixed point.

Proof Define a sequence, $\bar{U}_{n+1} = \mathcal{J}\bar{U}_n$ for $\bar{U}_0 \in \mathcal{Q}$. If $\bar{U}_{n_0} = \bar{U}_{n_0+1}$ for certain $n_0 \in \mathbb{N}$, then \bar{U}_{n_0} is a fixed point of \mathcal{J} . Assume that $\bar{U}_n \neq \bar{U}_{n+1}$ for each n . But $\bar{U}_0 \leq \mathcal{J}\bar{U}_0$ and \mathcal{J} is non-decreasing as by induction we obtain that

$$\bar{U}_0 \leq \bar{U}_1 \leq \bar{U}_2 \leq \dots \leq \bar{U}_n \leq \bar{U}_{n+1} \leq \dots \quad (2)$$

Now

$$\begin{aligned} & \varrho(\bar{U}_{n+1}, \bar{U}_n) \\ & = \varrho(\mathcal{J}\bar{U}_n, \mathcal{J}\bar{U}_{n-1}) \\ & \leq \alpha \frac{\varrho(\bar{U}_n, \mathcal{J}\bar{U}_n)\varrho(\bar{U}_{n-1}, \mathcal{J}\bar{U}_{n-1})}{\varrho(\bar{U}_n, \bar{U}_{n-1})} \\ & + \beta[\varrho(\bar{U}_n, \mathcal{J}\bar{U}_n) + \varrho(\bar{U}_{n-1}, \mathcal{J}\bar{U}_{n-1})] \\ & + \gamma\varrho(\bar{U}_n, \bar{U}_{n-1}) + \mathcal{L} \min\{\varrho(\bar{U}_n, \mathcal{J}\bar{U}_{n-1}), \varrho(\bar{U}_{n-1}, \mathcal{J}\bar{U}_n)\}, \end{aligned}$$

which infer that

$$\begin{aligned} \varrho(\bar{U}_{n+1}, \bar{U}_n) & \leq \left(\frac{\beta + \gamma}{1 - \alpha - \beta} \right) \varrho(\bar{U}_n, \bar{U}_{n-1}) \leq \dots \\ & \leq \left(\frac{\beta + \gamma}{1 - \alpha - \beta} \right)^n \varrho(\bar{U}_1, \bar{U}_0). \end{aligned}$$

Furthermore, the triangular inequality of d , we have for $m \geq n$,

$$\begin{aligned} \varrho(\bar{U}_n, \bar{U}_m) & = \varrho(\bar{U}_n, \bar{U}_{n+1}) + \varrho(\bar{U}_{n+1}, \bar{U}_{n+2}) \\ & + \dots + \varrho(\bar{U}_{m-1}, \bar{U}_m) \\ & \leq \left(\kappa^n + \kappa^{n+1} + \dots + \kappa^{m-1} \right) \varrho(\bar{U}_0, T\bar{U}_0) \\ & \leq \frac{\kappa^n}{1 - \kappa} \varrho(\bar{U}_1, \bar{U}_0), \end{aligned} \quad (3)$$

where $\kappa = \frac{\beta + \gamma}{1 - \alpha - \beta}$. As $n \rightarrow \infty$ in Eq. (3), we obtain $\varrho(\bar{U}_n, \bar{U}_m) = 0$. This shows that $\{\bar{U}_n\} \in \mathcal{Q}$ is a Cauchy sequence and then $\bar{U}_n \rightarrow \zeta \in \mathcal{Q}$ by its completeness. Besides, the continuity of \mathcal{J} implies that

$$\mathcal{J}\zeta = \mathcal{J} \left(\lim_{n \rightarrow \infty} \bar{U}_n \right) = \lim_{n \rightarrow \infty} T\bar{U}_n = \lim_{n \rightarrow \infty} \bar{U}_{n+1} = \zeta.$$

Therefore, ζ is a fixed point of \mathcal{J} in \mathcal{Q} . \square

Extracting the continuity of a map \mathcal{J} in Theorem 1, we have the below result.

Theorem 2 Suppose $(\mathcal{Q}, \varrho, \leq)$ is a complete partially ordered metric space. A non-decreasing mapping \mathcal{J} has a fixed point, if it satisfies the following assumption with $\bar{U}_0 \leq \mathcal{J} \bar{U}_0$ for certain $\bar{U}_0 \in \mathcal{Q}$.

If a nondecreasing sequence $\{\bar{U}_n\} \rightarrow \bar{U}$ in \mathcal{Q} ,
then $\bar{U} = \sup\{\bar{U}_n\}$. (4)

Proof The required proof can be obtained by following the proof of Theorem 8. \square

Example 1 Let $\mathcal{Q}_1 = \{(2, 0), (0, 2)\} \subseteq \mathbb{R}^2$ with the Euclidean distance ϱ . Define the partial order in \mathcal{Q}_1 as below

$(\bar{U}_1, \zeta_1) \leq (\bar{U}_2, \zeta_2)$ if and only if $\bar{U}_1 \leq \bar{U}_2$ and $\zeta_1 \leq \zeta_2$.

It is evident that, $(\mathcal{Q}_1, \varrho, \leq)$ is a complete partially ordered metric space and a map $\mathcal{J}(\bar{U}, \zeta) = (\bar{U}, \zeta)$ is non-decreasing and continuous. Consider

$$\begin{aligned} \varrho(\mathcal{J}(\bar{U}_1, \zeta_1), \mathcal{J}(\bar{U}_2, \zeta_2)) &\leq \gamma \varrho((\bar{U}_1, \zeta_1), (\bar{U}_2, \zeta_2)) \\ &\leq \alpha \frac{\varrho((\bar{U}_1, \zeta_1), \mathcal{J}(\bar{U}_1, \zeta_1)) \varrho((\bar{U}_2, \zeta_2), \mathcal{J}(\bar{U}_2, \zeta_2))}{\varrho((\bar{U}_1, \zeta_1), (\bar{U}_2, \zeta_2))} \\ &\quad + \beta [\varrho((\bar{U}_1, \zeta_1), \mathcal{J}(\bar{U}_1, \zeta_1)) + \varrho((\bar{U}_2, \zeta_2), \mathcal{J}(\bar{U}_2, \zeta_2))] \\ &\quad + \gamma \varrho((\bar{U}_1, \zeta_1), (\bar{U}_2, \zeta_2)) \\ &\quad + \mathcal{L} \min\{\varrho((\bar{U}_1, \zeta_1), \mathcal{J}(\bar{U}_2, \zeta_2)), \varrho((\bar{U}_2, \zeta_2), \mathcal{J}(\bar{U}_1, \zeta_1))\}, \end{aligned}$$

which holds for every $\alpha, \beta, \gamma \in [0, 1)$ with $0 \leq \alpha + 2\beta + \gamma < 1$ and any $\mathcal{L} \geq 0$. Also note that the elements of \mathcal{Q}_1 are comparable to themselves only. Furthermore, $(0, 2) \leq \mathcal{J}((0, 2))$. Therefore, all assumptions of Theorem 1 are met and \mathcal{J} has two fixed points $(2, 0)$, $(0, 2)$.

Example 2 The identity mapping \mathcal{J} has an infinite number of fixed points in $\mathcal{Q}_2 = \{(\bar{U}, -\bar{U}), \bar{U} \in \mathbb{R}\}$, as any two distinct elements are not comparable in \mathcal{Q}_2 with usual order and the Euclidean distance (ϱ).

Theorem 3 The unique fixed point of \mathcal{J} in Theorems 1 and 2 can be found from the condition (11) stated below.

Example 3 Define a self map $\mathcal{J} : \mathcal{Q} \rightarrow \mathcal{Q}$, where $\mathcal{Q} = [0, 1]$ with usual metric and usual order $\varepsilon \leq \bar{U}$, for $\bar{U}, \varepsilon \in \mathcal{Q}$ by

$$\varrho(\bar{U}, \varepsilon) = \begin{cases} \frac{\bar{U}}{24}, & \text{if } \bar{U} \in \left[0, \frac{1}{4}\right] \\ \frac{\bar{U}}{12} - \frac{1}{96}, & \text{if } \bar{U} \in \left(\frac{1}{4}, 1\right]. \end{cases}$$

Then \mathcal{J} has a unique fixed point in \mathcal{Q} .

Proof We will discuss the proof thoroughly by the subsequent cases.

Case: 1 If $\bar{U}, \varepsilon \in [0, \frac{1}{4})$, then

$$\begin{aligned} \varrho(\mathcal{J} \bar{U}, \mathcal{J} \varepsilon) &= \frac{1}{24} |\bar{U} - \varepsilon| \leq \frac{1}{10} |\bar{U} - \varepsilon| = \frac{1}{10} \varrho(\bar{U}, \varepsilon) \\ &\leq \alpha \frac{\varrho(\bar{U}, \mathcal{J} \bar{U}) \varrho(\varepsilon, \mathcal{J} \varepsilon)}{\varrho(\bar{U}, \varepsilon)} + \beta [\varrho(\bar{U}, \mathcal{J} \bar{U}) \\ &\quad + \varrho(\varepsilon, \mathcal{J} \varepsilon)] \\ &\quad + \frac{1}{10} \varrho(\bar{U}, \varepsilon) + \mathcal{L} \min\{\varrho(\bar{U}, \mathcal{J} \varepsilon), \\ &\quad \varrho(\varepsilon, \mathcal{J} \bar{U})\}, \end{aligned}$$

this inequality is true for every $\alpha, \beta \in [0, 1)$ with $0 \leq \alpha + 2\beta + \gamma < 1$, and $\mathcal{L} \geq 0$. Consequently all conditions of Theorem 1 are fulfilled in this case.

Case: 2 If $\bar{U}, \varepsilon \in (\frac{1}{4}, 1]$, then

$$\begin{aligned} \varrho(\mathcal{J} \bar{U}, \mathcal{J} \varepsilon) &= \frac{1}{12} |\bar{U} - \varepsilon| \leq \frac{1}{10} |\bar{U} - \varepsilon| \\ &= \frac{1}{10} \varrho(\bar{U}, \varepsilon) \\ &\leq \alpha \frac{\varrho(\bar{U}, \mathcal{J} \bar{U}) \varrho(\varepsilon, \mathcal{J} \varepsilon)}{\varrho(\bar{U}, \varepsilon)} \\ &\quad + \beta [\varrho(\bar{U}, \mathcal{J} \bar{U}) + \varrho(\varepsilon, \mathcal{J} \varepsilon)] \\ &\quad + \frac{1}{10} \varrho(\bar{U}, \varepsilon) \\ &\quad + \mathcal{L} \min\{\varrho(\bar{U}, \mathcal{J} \varepsilon), \varrho(\varepsilon, \mathcal{J} \bar{U})\}, \end{aligned}$$

this inequality holds for any $\mathcal{L} \geq 0$ and every $\alpha, \beta \in [0, 1)$ with $0 \leq \alpha + 2\beta + \gamma < 1$. Thus, all assumptions in Theorem 1 are met.

Case: 3 If $\varepsilon \in [0, \frac{1}{4})$ and $\bar{U} \in (\frac{1}{4}, 1]$, then we have $\frac{1}{96} |4\bar{U} - 1| \leq \frac{1}{96}$, $\frac{23}{96} \leq \varrho(\bar{U}, \mathcal{J} \varepsilon) = |\bar{U} - \frac{\varepsilon}{24}| \leq 1$, and $\frac{1}{96} \leq \varrho(\varepsilon, \mathcal{J} \bar{U}) = |\frac{\bar{U}}{12} - \frac{1}{96} - \varepsilon| \leq \frac{23}{96}$. Therefore,

$$\begin{aligned}
\varrho(\mathcal{I}\bar{U}, \mathcal{I}\varepsilon) &= \left| \frac{\bar{U}}{12} - \frac{1}{96} - \frac{\varepsilon}{24} \right| \\
&\leq \frac{1}{24}|\bar{U} - \varepsilon| + \frac{1}{96}|4\bar{U} - 1| \\
&\leq \alpha \frac{\varrho(\bar{U}, \mathcal{I}\bar{U})\varrho(\varepsilon, \mathcal{I}\varepsilon)}{\varrho(\bar{U}, \varepsilon)} + \beta[\varrho(\bar{U}, \mathcal{I}\bar{U}) \\
&\quad + \varrho(\varepsilon, \mathcal{I}\varepsilon)] \\
&\quad + \frac{1}{10}\varrho(\bar{U}, \varepsilon) + \mathcal{L} \min\{\varrho(\bar{U}, \mathcal{I}\bar{U}), \\
&\quad \varrho(\varepsilon, \mathcal{I}\varepsilon)\},
\end{aligned}$$

holds for any $\mathcal{L} \geq 0$ and for any $\alpha, \beta \in [0, 1)$ with $0 \leq \alpha + 2\beta + \gamma < 1$. Since all other hypotheses of Theorem 1 are satisfied, as a result $0 \in \mathcal{Q}$ is a unique fixed point of \mathcal{I} . \square

Corollary 1 Suppose $(\mathcal{Q}, \varrho, \preceq)$ is a complete partially ordered metric space. A non-decreasing continuous self-map \mathcal{I} on \mathcal{Q} satisfies

$$\begin{aligned}
\varrho(\mathcal{I}\bar{U}, \mathcal{I}\varepsilon) &\leq \alpha \frac{\varrho(\bar{U}, \mathcal{I}\bar{U})\varrho(\varepsilon, \mathcal{I}\varepsilon)}{\varrho(\bar{U}, \varepsilon)} \\
&\quad + \beta[\varrho(\bar{U}, \mathcal{I}\bar{U}) + \varrho(\varepsilon, \mathcal{I}\varepsilon)] + \gamma\varrho(\bar{U}, \varepsilon),
\end{aligned}$$

(5)

for any $\bar{U} \neq \varepsilon \in \mathcal{Q}$ with $\bar{U} \preceq \varepsilon$, and some $\alpha, \beta, \gamma \in [0, 1)$ with $0 \leq \alpha + 2\beta + \gamma < 1$. If $\bar{U}_0 \preceq \mathcal{I}\bar{U}_0$, for $\bar{U}_0 \in \mathcal{Q}$, then \mathcal{I} has a fixed point.

Proof Put $\mathcal{L} = 0$ in Theorem 1. \square

Example 4 Define a metric ϱ on $\mathcal{Q} = [0, \infty)$ by

$$\varrho(\bar{U}, \varepsilon) = \begin{cases} \max\{\bar{U}, \varepsilon\}, & \text{if } \bar{U} \neq \varepsilon \\ 0, & \text{if } \bar{U} = \varepsilon. \end{cases}$$

Also, let us define $\mathcal{I} : \mathcal{Q} \rightarrow \mathcal{Q}$ by

$$\mathcal{I}\bar{U} = \begin{cases} \frac{\bar{U}}{10(1+\bar{U})}, & \text{if } 0 \leq \bar{U} \leq 5, \\ \frac{\bar{U}}{20}, & \text{if } 5 < \bar{U}, \end{cases}$$

with $\bar{U} \leq \varepsilon$ iff $\bar{U} \leq \varepsilon$. Then from Corollary 1, \mathcal{I} has a fixed point.

Proof Consider the subsequent attainable cases to debate the proof of the theory.

Case: 1 If $0 \leq \bar{U} < \varepsilon \leq 5$, then

$$\begin{aligned}
\varrho(\mathcal{I}\bar{U}, \mathcal{I}\varepsilon) &= \max\{\mathcal{I}\bar{U}, \mathcal{I}\varepsilon\} = \max\left\{\frac{\bar{U}}{10(1+\bar{U})}, \frac{\varepsilon}{10(1+\varepsilon)}\right\} \leq \frac{2}{5}\varepsilon \\
&= \frac{1}{5}(\bar{U} + \varepsilon) + \frac{1}{5}(\bar{U} + \varepsilon) = \frac{1}{5}\left(\frac{\bar{U}\varepsilon}{\varepsilon} + \varepsilon\right) + \frac{1}{5}(\bar{U} + \varepsilon) \\
&= \frac{1}{5}\left[\frac{\max\left\{\bar{U}, \frac{\bar{U}}{10(1+\bar{U})}\right\}, \max\left\{\varepsilon, \frac{\varepsilon}{10(1+\varepsilon)}\right\}}{\max\{\bar{U}, \varepsilon\}}\right] \\
&\quad + \left(\max\left\{\bar{U}, \frac{\bar{U}}{10(1+\bar{U})}\right\} + \max\left\{\varepsilon, \frac{\varepsilon}{10(1+\varepsilon)}\right\}\right) + \max\{\bar{U}, \varepsilon\} \\
&= \frac{1}{5}\left(\frac{\varrho(\bar{U}, \mathcal{I}\bar{U})\varrho(\varepsilon, \mathcal{I}\varepsilon)}{\varrho(\bar{U}, \varepsilon)} + [\varrho(\bar{U}, \mathcal{I}\bar{U}) + \varrho(\varepsilon, \mathcal{I}\varepsilon)] + \varrho(\bar{U}, \varepsilon)\right),
\end{aligned}$$

implies that,

$$\begin{aligned} \varrho(\mathcal{J} \bar{U}, \mathcal{J} \varepsilon) &\leq \frac{1}{5} \frac{\varrho(\bar{U}, \mathcal{J} \bar{U}) \varrho(\varepsilon, \mathcal{J} \varepsilon)}{\varrho(\bar{U}, \varepsilon)} \\ &\quad + \frac{1}{5} [\varrho(\bar{U}, \mathcal{J} \bar{U}) + \varrho(\varepsilon, \mathcal{J} \varepsilon)] + \frac{1}{5} \varrho(\bar{U}, \varepsilon). \end{aligned}$$

Case: 2 If $5 < \bar{U} < \varepsilon$, then

Apart from, if \mathcal{Q} satisfies the conditions (4) and (11), then a mapping \mathcal{J} has a fixed point and also its uniqueness in Corollary 1.

Theorem 4 Suppose $(\mathcal{Q}, \varrho, \preceq)$ is a complete partially ordered metric space. A non-decreasing self mapping \mathcal{J} is such that either \mathcal{J} is continuous or \mathcal{Q} satisfies the following

$$\begin{aligned} \varrho(\mathcal{J} \bar{U}, \mathcal{J} \varepsilon) &= \max\{\mathcal{J} \bar{U}, \mathcal{J} \varepsilon\} = \max\left\{\frac{\bar{U}}{20}, \frac{\varepsilon}{20}\right\} = \frac{\varepsilon}{20} \leq \frac{2}{5} \varepsilon \\ &= \frac{1}{5}(\bar{U} + \varepsilon) + \frac{1}{5}(\bar{U} + \varepsilon) = \frac{1}{5}\left(\frac{\bar{U}\varepsilon}{\varepsilon} + \varepsilon\right) + \frac{1}{5}(\bar{U} + \varepsilon) \\ &= \frac{1}{5} \left[\frac{\max\left\{\bar{U}, \frac{\bar{U}}{20}\right\}, \max\left\{\varepsilon, \frac{\varepsilon}{20}\right\}}{\max\{\bar{U}, \varepsilon\}} + \left[\max\left\{\bar{U}, \frac{\bar{U}}{20}\right\} + \max\left\{\varepsilon, \frac{\varepsilon}{20}\right\} \right] + \max\{\bar{U}, \varepsilon\} \right] \\ &= \frac{1}{5} \left(\frac{\varrho(\bar{U}, \mathcal{J} \bar{U}) \varrho(\varepsilon, \mathcal{J} \varepsilon)}{\varrho(\bar{U}, \varepsilon)} + [\varrho(\bar{U}, \mathcal{J} \bar{U}) + \varrho(\varepsilon, \mathcal{J} \varepsilon)] + \varrho(\bar{U}, \varepsilon) \right), \end{aligned}$$

which implies that,

$$\begin{aligned} \varrho(\mathcal{J} \bar{U}, \mathcal{J} \varepsilon) &\leq \frac{1}{5} \frac{\varrho(\bar{U}, \mathcal{J} \bar{U}) \varrho(\varepsilon, \mathcal{J} \varepsilon)}{\varrho(\bar{U}, \varepsilon)} \\ &\quad + \frac{1}{5} [\varrho(\bar{U}, \mathcal{J} \bar{U}) + \varrho(\varepsilon, \mathcal{J} \varepsilon)] + \frac{1}{5} \varrho(\bar{U}, \varepsilon). \end{aligned}$$

Case: 3 If $0 \leq \bar{U} \leq 5$ and $5 < \varepsilon$, then

condition in Theorems 1 and 2 and Corollary 1, then \mathcal{J} has a fixed point in \mathcal{Q} , for $\bar{U}_0 \in \mathcal{Q}$ such that $\bar{U}_0 \succeq \mathcal{J} \bar{U}_0$.

If a nonincreasing sequence $\{\bar{U}_n\} \rightarrow \bar{U}$ in \mathcal{Q} , then $\bar{U} = \inf\{\bar{U}_n\}$.

Proof The scheme of the proof is similar to the procedure of the proofs of the previous theorems. \square

$$\begin{aligned} \varrho(\mathcal{J} \bar{U}, \mathcal{J} \varepsilon) &= \max\{\mathcal{J} \bar{U}, \mathcal{J} \varepsilon\} = \max\left\{\frac{\bar{U}}{10(1+\bar{U})}, \frac{\varepsilon}{20}\right\} = \frac{\varepsilon}{20} \leq \frac{2}{5} \varepsilon \\ &= \frac{1}{5}(\bar{U} + \varepsilon) + \frac{1}{5}(\bar{U} + \varepsilon) = \frac{1}{5}\left(\frac{\bar{U}\varepsilon}{\varepsilon} + \varepsilon\right) + \frac{1}{5}(\bar{U} + \varepsilon) \\ &= \frac{1}{5} \left[\frac{\max\left\{\bar{U}, \frac{\bar{U}}{10(1+\bar{U})}\right\}, \max\left\{\varepsilon, \frac{\varepsilon}{20}\right\}}{\max\{\bar{U}, \varepsilon\}} + \left[\max\left\{\bar{U}, \frac{\bar{U}}{10(1+\bar{U})}\right\} + \max\left\{\varepsilon, \frac{\varepsilon}{20}\right\} \right] + \max\{\bar{U}, \varepsilon\} \right] \\ &= \frac{1}{5} \left(\frac{\varrho(\bar{U}, \mathcal{J} \bar{U}) \varrho(\varepsilon, \mathcal{J} \varepsilon)}{\varrho(\bar{U}, \varepsilon)} + [\varrho(\bar{U}, \mathcal{J} \bar{U}) + \varrho(\varepsilon, \mathcal{J} \varepsilon)] + \varrho(\bar{U}, \varepsilon) \right). \end{aligned}$$

Therefore,

$$\begin{aligned} \varrho(\mathcal{J} \bar{U}, \mathcal{J} \varepsilon) &\leq \frac{1}{5} \frac{\varrho(\bar{U}, \mathcal{J} \bar{U}) \varrho(\varepsilon, \mathcal{J} \varepsilon)}{\varrho(\bar{U}, \varepsilon)} \\ &\quad + \frac{1}{5} [\varrho(\bar{U}, \mathcal{J} \bar{U}) + \varrho(\varepsilon, \mathcal{J} \varepsilon)] + \frac{1}{5} \varrho(\bar{U}, \varepsilon). \end{aligned}$$

Subsequently, all conditions of Corollary 1 are fulfilled and hence the self mapping \mathcal{J} has a fixed point $0 \in \mathcal{Q}$. \square

In particular, there is an example where Theorem 1 (or Corollary 1) can be applied and not be valid in a complete metric space.

Example 5 Let $\mathcal{Q} = \{(0, 1), (1, 0), (1, 1)\}$ and, let the partial order relation on \mathcal{Q} be $R = \{(\bar{U}, \bar{U}) : \bar{U} \in \mathcal{Q}\}$. Observe that the elements only in \mathcal{Q} are comparable to themselves. Apart from, (\mathcal{Q}, ϱ) is a complete metric space with the Euclidean distance (ϱ) while with regards \preceq is a partially ordered set.

Define a map $\mathcal{J} : \mathcal{Q} \rightarrow \mathcal{Q}$ by

$$\mathcal{J}(0, 1) = (1, 0), \quad \mathcal{J}(1, 0) = (0, 1), \quad \mathcal{J}(1, 1) = (1, 1),$$

is a nondecreasing, continuous and, $(1, 1) \leq \mathcal{J}(1, 1) = (1, 1)$ for $(1, 1) \in \mathcal{Q}$ and satisfy condition (1) (or(5)). As a result $(1, 1)$ is a fixed point of \mathcal{J} .

Besides, for $\mathcal{U} = (0, 1)$, $\zeta = (1, 0)$ in \mathcal{Q} , we have

$$\begin{aligned} \varrho(\mathcal{J}\mathcal{U}, \mathcal{J}\zeta) &= \sqrt{2}, \quad \varrho(\mathcal{U}, \mathcal{J}\zeta) = 0, \quad \varrho(\zeta, \mathcal{J}\mathcal{U}) = 0, \\ \varrho(\mathcal{U}, \mathcal{J}\mathcal{U}) &= \sqrt{2}, \quad \varrho(\zeta, \mathcal{J}\zeta) = \sqrt{2}, \end{aligned}$$

then

$$\begin{aligned} \varrho(\mathcal{J}\mathcal{U}, \mathcal{J}\zeta) &= \sqrt{2} \leq \alpha \frac{\varrho(\mathcal{U}, \mathcal{J}\mathcal{U}) \varrho(\zeta, \mathcal{J}\zeta)}{\varrho(\mathcal{U}, \zeta)} + \beta [\varrho(\mathcal{U}, \mathcal{J}\mathcal{U}) \\ &\quad + \varrho(\zeta, \mathcal{J}\zeta)] + \gamma \varrho(\mathcal{U}, \zeta) \\ &\leq \alpha \cdot \frac{\sqrt{2} \cdot \sqrt{2}}{\sqrt{2}} + \beta [\sqrt{2} + \sqrt{2}] + \gamma \cdot \sqrt{2} \\ &= (\alpha + 2\beta + \gamma) \cdot \sqrt{2}, \end{aligned}$$

which implies that, $\alpha + 2\beta + \gamma \geq 1$. Accordingly, this example is not valid in the case of usual complete metrical space.

Also, notice here that \mathcal{J} has a unique fixed point even though \mathcal{Q} doesn't satisfies the condition (11) stated below. Hence, as a result condition (11) is not necessary for the existence of the uniqueness of a fixed point.

In the next theorem, we set up the existence of a unique fixed point of a mapping \mathcal{J} through assuming most effective the continuity of some iteration of it.

Theorem 5 *If \mathcal{J}^p is continuous for some positive integer p in Theorem 1, then \mathcal{J} has a fixed point.*

Proof From Theorem 1, there is a Cauchy sequence $\{\mathcal{U}_n\} \subset \mathcal{Q}$ such that $\{\mathcal{U}_n\} \rightarrow \zeta \in \mathcal{Q}$ as a result its subsequence $\mathcal{U}_{n_k} (n_k = kp)$ converges to the same point. Moreover,

$$\mathcal{J}^p \zeta = \mathcal{J}^p \left(\lim_{n \rightarrow \infty} \mathcal{U}_{n_k} \right) = \lim_{n \rightarrow \infty} \mathcal{U}_{n_{k+1}} = \zeta,$$

which shows that ζ is a fixed point of \mathcal{J}^p . Next to claim that $\mathcal{J}\zeta = \zeta$. Assume m is the smallest among all positive integer so that $\mathcal{J}^m \zeta = \zeta$ and $\mathcal{J}^q \zeta \neq \zeta$ ($q = 1, 2, 3, \dots, m-1$). If $m > 1$, then

$$\begin{aligned} \varrho(\mathcal{J}\zeta, \zeta) &= \varrho(\mathcal{J}\zeta, \mathcal{J}^m \zeta) \\ &\leq \alpha \frac{\varrho(\zeta, \mathcal{J}\zeta) \varrho(\mathcal{J}^{m-1} \zeta, \mathcal{J}^m \zeta)}{\varrho(\zeta, \mathcal{J}^{m-1} \zeta)} \\ &\quad + \beta [\varrho(\zeta, \mathcal{J}\zeta) \\ &\quad + \varrho(\mathcal{J}^{m-1} \zeta, \mathcal{J}^m \zeta)] + \gamma \varrho(\zeta, \mathcal{J}^{m-1} \zeta) \\ &\quad + \mathcal{L} \min\{\varrho(\zeta, \mathcal{J}^m \zeta), \\ &\quad \varrho(\mathcal{J}^{m-1} \zeta, \mathcal{J}\zeta)\}. \end{aligned}$$

Therefore,

$$\varrho(\zeta, \mathcal{J}\zeta) \leq \left(\frac{\beta + \gamma}{1 - \alpha - \beta} \right) \varrho(\zeta, \mathcal{J}^{m-1} \zeta).$$

Regarding (1), we have

$$\begin{aligned} \varrho(\zeta, \mathcal{J}^{m-1} \zeta) &= \varrho(\mathcal{J}^m \zeta, \mathcal{J}^{m-1} \zeta) \\ &\leq \alpha \frac{\varrho(\mathcal{J}^{m-1} \zeta, \mathcal{J}^m \zeta) \cdot \varrho(\mathcal{J}^{m-2} \zeta, \mathcal{J}^{m-1} \zeta)}{\varrho(\mathcal{J}^{m-1} \zeta, \mathcal{J}^{m-2} \zeta)} \\ &\quad + \beta [\varrho(\mathcal{J}^{m-1} \zeta, \mathcal{J}^m \zeta) + \varrho(\mathcal{J}^{m-2} \zeta, \mathcal{J}^{m-1} \zeta)] + \gamma \varrho(\mathcal{J}^{m-1} \zeta, \mathcal{J}^{m-2} \zeta) \\ &\quad + \mathcal{L} \min\{\varrho(\mathcal{J}^{m-1} \zeta, \mathcal{J}^{m-1} \zeta), \varrho(\mathcal{J}^{m-2} \zeta, \mathcal{J}^m \zeta)\}. \end{aligned}$$

By induction, we get

$$\begin{aligned}\varrho(\zeta, \mathcal{J}^{m-1}\zeta) &= \varrho(\mathcal{J}^m\zeta, \mathcal{J}^{m-1}\zeta) \\ &\leq \kappa\varrho(\mathcal{J}^{m-1}\zeta, \mathcal{J}^{m-2}\zeta) \\ &\leq \dots \leq \kappa^{m-1}\varrho(\mathcal{J}\zeta, \zeta),\end{aligned}$$

where $\kappa = \frac{\beta+\gamma}{1-\alpha-\beta} < 1$. Therefore,

$$\varrho(\mathcal{J}\zeta, \zeta) \leq \kappa^m\varrho(\mathcal{J}\zeta, \zeta) < \varrho(\mathcal{J}\zeta, \zeta),$$

a contradiction. Hence, $\mathcal{J}\zeta = \zeta$. \square

Corollary 2 If \mathcal{J}^p is continuous for some positive integer p , then \mathcal{J} has a fixed point in Corollary 1.

Proof Put $\mathcal{L} = 0$ in Theorem 5. \square

Theorem 6 Suppose $(\mathcal{Q}, \varrho, \leq)$ is a complete partially ordered metric space and \mathcal{J} be a non-decreasing self map on \mathcal{Q} . Assume for some positive integer m , \mathcal{J} satisfies

$$\begin{aligned}\varrho(\mathcal{J}^m\mathcal{U}, \mathcal{J}^m\varepsilon) &\leq \alpha \frac{\varrho(\mathcal{U}, \mathcal{J}^m\mathcal{U})\varrho(\varepsilon, \mathcal{J}^m\varepsilon)}{\varrho(\mathcal{U}, \varepsilon)} \\ &\quad + \beta[\varrho(\mathcal{U}, \mathcal{J}^m\mathcal{U}) + \varrho(\varepsilon, \mathcal{J}^m\varepsilon)] \\ &\quad + \gamma\varrho(\mathcal{U}, \varepsilon) + \mathcal{L} \min\{\varrho(\mathcal{U}, \mathcal{J}^m\varepsilon), \\ &\quad \varrho(\varepsilon, \mathcal{J}^m\mathcal{U})\},\end{aligned}\tag{6}$$

for any $\mathcal{U} \neq \varepsilon \in \mathcal{Q}$ with $\mathcal{U} \leq \varepsilon$, where $\mathcal{L} \geq 0$, and $\alpha, \beta, \gamma \in [0, 1)$ with $0 \leq \alpha + 2\beta + \gamma < 1$. If $\mathcal{U}_0 \leq \mathcal{J}^m\mathcal{U}_0$

$$\varrho(\mathcal{J}\mathcal{U}, \mathcal{J}\varepsilon) \leq \begin{cases} \lambda\varrho(\mathcal{U}, \varepsilon) + \theta[\varrho(\mathcal{U}, \mathcal{J}\mathcal{U}) + \varrho(\varepsilon, \mathcal{J}\varepsilon)] \\ \quad + \mu \frac{\varrho(\mathcal{U}, \mathcal{J}\mathcal{U})\varrho(\mathcal{U}, \mathcal{J}\varepsilon) + \varrho(\varepsilon, \mathcal{J}\mathcal{U})\varrho(\varepsilon, \mathcal{J}\varepsilon)}{\varrho(\varepsilon, \mathcal{J}\mathcal{U}) + \varrho(\mathcal{U}, \mathcal{J}\varepsilon)} \end{cases}, \text{ if } A \neq 0$$

for certain $\mathcal{U}_0 \in \mathcal{Q}$ and \mathcal{J}^m is continuous, then \mathcal{J} has a fixed point.

Proof The proof follows Theorems 1 and 5. \square

Corollary 3 Let $(\mathcal{Q}, \varrho, \leq)$ be a complete partially ordered metric space. A self map \mathcal{J} has a fixed point, if $\mathcal{U}_0 \leq \mathcal{J}^m\mathcal{U}_0$ for certain $\mathcal{U}_0 \in \mathcal{Q}$ and satisfies the below contraction condition for some positive integer m ,

$$\begin{aligned}\varrho(\mathcal{J}^m\mathcal{U}, \mathcal{J}^m\varepsilon) &\leq \alpha \frac{\varrho(\mathcal{U}, \mathcal{J}^m\mathcal{U})\varrho(\varepsilon, \mathcal{J}^m\varepsilon)}{\varrho(\mathcal{U}, \varepsilon)} \\ &\quad + \beta[\varrho(\mathcal{U}, \mathcal{J}^m\mathcal{U}) + \varrho(\varepsilon, \mathcal{J}^m\varepsilon)] + \gamma\varrho(\mathcal{U}, \varepsilon),\end{aligned}\tag{7}$$

for all $\mathcal{U} \neq \varepsilon \in \mathcal{Q}$ with $\mathcal{U} \leq \varepsilon$, and for some $\alpha, \beta, \gamma \in [0, 1)$ such that $0 \leq \alpha + 2\beta + \gamma < 1$.

Proof Setting $\mathcal{L} = 0$ in Theorem 6, the required proof can be found. \square

Let us see the example below.

Example 6 Let $\mathcal{Q} = [0, 1]$ with the usual metric and usual order \leq . Define a map $\mathcal{J} : \mathcal{Q} \rightarrow \mathcal{Q}$ by

$$\mathcal{J}\mathcal{U} = \begin{cases} 0, & \text{if } \mathcal{U} \in [0, \frac{1}{6}], \\ \frac{1}{6}, & \text{if } \mathcal{U} \in (\frac{1}{6}, 1], \end{cases}$$

then \mathcal{J} is discontinuous and is not satisfying condition (1) for each $\alpha, \beta, \gamma \in [0, 1)$ with $0 \leq \alpha + 2\beta + \gamma < 1$ where as $\mathcal{U} = \frac{1}{6}, \varepsilon = 1$. But $\mathcal{J}^2(\mathcal{U}) = 0$ for all $\mathcal{U} \in [0, 1]$ and \mathcal{J}^2 fulfill all assumptions of Theorem 6. Therefore, \mathcal{J}^2 has a unique fixed point $0 \in \mathcal{Q}$.

Generalized Rational Contraction Results

Theorem 7 Suppose $(\mathcal{Q}, \varrho, \leq)$ is a complete partially ordered metric space. A non-decreasing continuous self map \mathcal{J} on \mathcal{Q} satisfies

for any $\mathcal{U} \neq \varepsilon \in \mathcal{Q}$ with $\varepsilon \leq \mathcal{U}$, where $A = \varrho(\varepsilon, \mathcal{J}\mathcal{U}) + \varrho(\mathcal{U}, \mathcal{J}\varepsilon)$ and, λ, θ, μ are non-negative reals such that $0 \leq \lambda + 2\theta + \mu < 1$. If $\mathcal{U}_0 \leq \mathcal{J}\mathcal{U}_0$ for certain $\mathcal{U}_0 \in \mathcal{Q}$, then \mathcal{J} has a fixed point.

Proof The proof is trivial, if $\mathcal{U}_0 = \mathcal{J}\mathcal{U}_0$. Suppose not, $\mathcal{U}_0 < \mathcal{J}\mathcal{U}_0$ and then the non-decreasing property of \mathcal{J} , we acquire that

$$\mathcal{U}_0 < \mathcal{J}\mathcal{U}_0 \leq \mathcal{J}^2\mathcal{U}_0 \leq \dots \leq \mathcal{J}^n\mathcal{U}_0 \leq \mathcal{J}^{n+1}\mathcal{U}_0 \leq \dots.\tag{9}$$

If $\mathcal{U}_{n_0} = \mathcal{U}_{n_0+1}$ for certain $n_0 \in \mathbb{N}$, then \mathcal{U}_{n_0} is a fixed point of \mathcal{J} from (9). Assume, $\mathcal{U}_n \neq \mathcal{U}_{n+1}$ ($n \geq 0$). From

(9), \mathcal{U}_n and \mathcal{U}_{n-1} are comparable for each $n \in \mathbb{N}$ then we have the discussion below in subsequent cases.

Case 1: If $A = \varrho(\mathcal{U}_{n-1}, \mathcal{J} \mathcal{U}_n) + \varrho(\mathcal{U}_n, \mathcal{J} \mathcal{U}_{n-1}) \neq 0$, then (8) implies that,

$$\begin{aligned} \varrho(\mathcal{U}_{n+1}, \mathcal{U}_n) &= \varrho(\mathcal{J} \mathcal{U}_n, \mathcal{J} \mathcal{U}_{n-1}) \\ &\leq \lambda \varrho(\mathcal{U}_n, \mathcal{U}_{n-1}) + \theta [\varrho(\mathcal{U}_n, \mathcal{J} \mathcal{U}_n) + \varrho(\mathcal{U}_{n-1}, \mathcal{J} \mathcal{U}_{n-1})] \\ &\quad + \mu \frac{\varrho(\mathcal{U}_n, \mathcal{J} \mathcal{U}_n) \varrho(\mathcal{U}_n, \mathcal{J} \mathcal{U}_{n-1}) + \varrho(\mathcal{U}_{n-1}, \mathcal{J} \mathcal{U}_n) \varrho(\mathcal{U}_{n-1}, \mathcal{J} \mathcal{U}_{n-1})}{\varrho(\mathcal{U}_{n-1}, \mathcal{J} \mathcal{U}_n) + \varrho(\mathcal{U}_n, \mathcal{J} \mathcal{U}_{n-1})}, \end{aligned}$$

which implies that,

$$\begin{aligned} \varrho(\mathcal{U}_{n+1}, \mathcal{U}_n) &\leq \lambda \varrho(\mathcal{U}_n, \mathcal{U}_{n-1}) + \theta [\varrho(\mathcal{U}_n, \mathcal{U}_{n+1}) + \varrho(\mathcal{U}_{n-1}, \mathcal{U}_n)] \\ &\quad + \mu \frac{\varrho(\mathcal{U}_n, \mathcal{U}_{n+1}) \varrho(\mathcal{U}_n, \mathcal{U}_n) + \varrho(\mathcal{U}_{n-1}, \mathcal{U}_{n+1}) \varrho(\mathcal{U}_{n-1}, \mathcal{U}_n)}{\varrho(\mathcal{U}_{n-1}, \mathcal{U}_{n+1}) + \varrho(\mathcal{U}_n, \mathcal{U}_n)}. \end{aligned}$$

Thus,

$$\begin{aligned} \varrho(\mathcal{U}_{n+1}, \mathcal{U}_n) &\leq \lambda \varrho(\mathcal{U}_n, \mathcal{U}_{n-1}) \\ &\quad + \theta [\varrho(\mathcal{U}_n, \mathcal{U}_{n+1}) + \varrho(\mathcal{U}_{n-1}, \mathcal{U}_n)] + \mu \varrho(\mathcal{U}_{n-1}, \mathcal{U}_n). \end{aligned}$$

Hence,

$$\varrho(\mathcal{U}_{n+1}, \mathcal{U}_n) \leq \left(\frac{\lambda + \theta + \mu}{1 - \theta} \right) \varrho(\mathcal{U}_{n-1}, \mathcal{U}_n).$$

Inductively, we get

$$\varrho(\mathcal{U}_{n+1}, \mathcal{U}_n) \leq \hbar^n \varrho(\mathcal{U}_1, \mathcal{U}_0),$$

here $\hbar = \frac{\lambda + \theta + \mu}{1 - \theta} < 1$. Also, by the triangular inequality of d , for $m \geq n$

$$\begin{aligned} \varrho(\mathcal{U}_m, \mathcal{U}_n) &\leq \varrho(\mathcal{U}_m, \mathcal{U}_{m-1}) + \varrho(\mathcal{U}_{m-1}, \mathcal{U}_{m-2}) \\ &\quad \dots + \varrho(\mathcal{U}_{n+1}, \mathcal{U}_n) \\ &\leq \frac{\hbar^n}{1 - \hbar} \varrho(\mathcal{U}_1, \mathcal{U}_0), \end{aligned}$$

which implies that, $\varrho(\mathcal{U}_m, \mathcal{U}_n) \rightarrow 0$ as $m, n \rightarrow \infty$. Thus, $\{\mathcal{U}_n\} \subset \mathcal{Q}$ is a Chachy sequence and converges to $\zeta \in \mathcal{Q}$. Besides, the continuity of \mathcal{J} gives that,

$$\begin{aligned} \mathcal{J} \zeta &= \mathcal{J} \left(\lim_{n \rightarrow \infty} \mathcal{U}_n \right) \\ &= \lim_{n \rightarrow \infty} \mathcal{J} \mathcal{U}_n = \lim_{n \rightarrow \infty} \mathcal{U}_{n+1} = \zeta. \end{aligned}$$

Therefore, $\zeta \in \mathcal{Q}$ is a fixed point of \mathcal{J} .

Case 2: If $A = \varrho(\mathcal{U}_{n-1}, \mathcal{J} \mathcal{U}_n) + \varrho(\mathcal{U}_n, \mathcal{J} \mathcal{U}_{n-1}) = 0$, then $\varrho(\mathcal{U}_{n+1}, \mathcal{U}_n) = 0$. As a result $\mathcal{U}_n = \mathcal{U}_{n+1}$, which is a contradiction. Hence, a fixed point $\zeta \in \mathcal{Q}$ for \mathcal{J} exists. \square

Example 7 Let us define a self map \mathcal{J} on $\mathcal{Q} = [0, 1]$ with usual metric and usual order \leq as

$$\mathcal{J} \mathcal{U} = \frac{5}{16(\mathcal{U}^2 + \mathcal{U} + \frac{15}{16})}.$$

Then \mathcal{J} has a fixed point in \mathcal{Q} .

Proof It is evident that \mathcal{J} is continuous and non-decreasing in $\mathcal{Q} = [0, 1]$ and $\mathcal{U}_0 = 0 \in \mathcal{Q}$ such that $\mathcal{U}_0 = 0 \leq \mathcal{J} \mathcal{U}_0$. For, $\mathcal{U} \leq \varepsilon$,

$$\begin{aligned} \varrho(\mathcal{J} \mathcal{U}, \mathcal{J} \varepsilon) &= \frac{5}{16} \left| \frac{1}{\mathcal{U}^2 + \mathcal{U} + \frac{15}{16}} - \frac{1}{\varepsilon^2 + \varepsilon + \frac{15}{16}} \right| \\ &= \frac{5}{16} \left| \frac{(\varepsilon + \mathcal{U})(\varepsilon - \mathcal{U}) + (\varepsilon - \mathcal{U})}{(\mathcal{U}^2 + \mathcal{U} + \frac{15}{16})(\varepsilon^2 + \varepsilon + \frac{15}{16})} \right| \\ &= \left| \frac{5(\mathcal{U} + \varepsilon + 1)}{16(\mathcal{U}^2 + \mathcal{U} + \frac{15}{16})(\varepsilon^2 + \varepsilon + \frac{15}{16})} \right| |\varepsilon - \mathcal{U}| \\ &\leq \frac{16}{45} |\varepsilon - \mathcal{U}|, \end{aligned}$$

holds for every $\mathcal{U}, \varepsilon \in \mathcal{Q}$. For $\lambda = \frac{16}{45}$ and $\theta, \mu \in [0, 1)$ such that $0 \leq \lambda + 2\theta + \mu < 1$, then $\frac{1}{4} \in \mathcal{Q}$ is a fixed point of \mathcal{J} as all the conditions of Theorem 7 are satisfied. \square

Extracting the continuity criteria on \mathcal{J} in Theorem 7, we have the following result.

Theorem 8 If \mathcal{Q} has an ordered complete(OC) property in Theorem 7, then a non-decreasing mapping \mathcal{J} has a fixed point in \mathcal{Q} .

Proof We only claim that $\zeta = \mathcal{J}\zeta$. By an ordered complete metrical property of \mathcal{Q} , we have $\zeta = \sup\{\mathcal{U}_n\}$, for $n \in \mathbb{N}$ as $\mathcal{U}_n \rightarrow \zeta \in \mathcal{Q}$ is a non-decreasing sequence. The non-decreasing property of a map \mathcal{J} implies that $\mathcal{J}\mathcal{U}_n \leq \mathcal{J}\zeta$ or, equivalently, $\mathcal{U}_{n+1} \leq \mathcal{J}\zeta$, for $n \geq 0$. Since, $\mathcal{U}_0 < \mathcal{U}_1 \leq \mathcal{J}\zeta$ and $\zeta = \sup\{\mathcal{U}_n\}$ as a result, we get $\zeta \leq \mathcal{J}\zeta$.

Assume $\zeta < \mathcal{J}\zeta$. From Theorem 7, there is a non-decreasing sequence $\mathcal{J}^n \zeta \in \mathcal{Q}$ with $\lim_{n \rightarrow \infty} \mathcal{J}^n \zeta = \varepsilon \in \mathcal{Q}$. Again by an ordered complete (OC) property of \mathcal{Q} , we obtain that $\varepsilon = \sup\{\mathcal{J}^n \zeta\}$. Furthermore, $\mathcal{U}_n = T^n \mathcal{U}_0 \leq \mathcal{J}^n \zeta$, for $n \geq 1$ as a result, $\mathcal{U}_n < \mathcal{J}^n \zeta$, $n \geq 1$, since $\mathcal{U}_n \leq \zeta < \mathcal{J}\zeta \leq \mathcal{J}^n \zeta$, for $n \geq 1$ whereas \mathcal{U}_n and $\mathcal{J}^n \zeta$, for $n \geq 1$ are distinct and comparable.

$n \rightarrow \infty$. By following the similar argument in Case 1, we get $\mathcal{J}\zeta = \zeta$. \square

Now, found some examples below where there is no assurance of a unique fixed point in Theorems 7 and 8.

Example 8 Let $\mathcal{Q} = \{(1, 0), (0, 1)\} \subseteq \mathbb{R}^2$ with the Euclidean distance (ϱ). Define a partial order (\mathcal{U}) in \mathcal{Q} as below:

$$\mathcal{U} : (m, n) \leq (p, q) \text{ if and only if } m \leq p \text{ and } n \leq q.$$

Let $\mathcal{J} : \mathcal{Q} \rightarrow \mathcal{Q}$ by $\mathcal{J}(\mathcal{U}, \zeta) = (\mathcal{U}, \zeta)$. Then \mathcal{J} have fixed points in \mathcal{Q} .

Proof It's obvious that, $(\mathcal{Q}, \varrho, \leq)$ is a complete partially ordered metric space and also, \mathcal{J} is a continuous and non-decreasing mapping satisfying

$$\begin{aligned} \varrho(\mathcal{J}(m, n), \mathcal{J}(p, q)) &\leq \lambda \varrho((m, n), (p, q)) \\ &\leq \lambda \varrho((m, n), (p, q)) + \theta [\varrho((m, n), \mathcal{J}(m, n)) + \varrho((p, q), \mathcal{J}(p, q))] \\ &\quad + \mu \frac{\varrho((m, n), \mathcal{J}(m, n)) \varrho((m, n), \mathcal{J}(p, q)) + \varrho((p, q), \mathcal{J}(m, n)) \varrho((p, q), \mathcal{J}(p, q))}{\varrho((p, q), \mathcal{J}(m, n)) + \varrho((m, n), \mathcal{J}(p, q))}, \end{aligned}$$

Now we have the discussion below in the subsequent cases.

Case 1: If $\varrho(\mathcal{J}^n \zeta, \mathcal{J}\mathcal{U}_n) + \varrho(\mathcal{U}_n, \mathcal{J}^{n+1} \zeta) \neq 0$, then Eq. (8) becomes,

$$\begin{aligned} &\varrho(\mathcal{U}_{n+1}, \mathcal{J}^{n+1} \zeta) \\ &= \varrho(\mathcal{J}\mathcal{U}_n, \mathcal{J}(\mathcal{J}^n \zeta)) \\ &\leq \lambda \varrho(\mathcal{U}_n, \mathcal{J}^n \zeta) \\ &\quad + \theta [\varrho(\mathcal{U}_n, \mathcal{U}_{n+1}) + \varrho(\mathcal{J}^n \zeta, \mathcal{J}^{n+1} \zeta)] \\ &\quad + \mu \frac{\varrho(\mathcal{U}_n, \mathcal{U}_{n+1}) \varrho(\mathcal{U}_n, \mathcal{J}^{n+1} \zeta) + \varrho(\mathcal{J}^n \zeta, \mathcal{U}_{n+1}) \varrho(\mathcal{J}^n \zeta, \mathcal{J}^{n+1} \zeta)}{\varrho(\mathcal{J}^n \zeta, \mathcal{U}_{n+1}) + \varrho(\mathcal{U}_n, \mathcal{J}^{n+1} \zeta)}. \end{aligned} \quad (10)$$

As $n \rightarrow \infty$ in Eq. (10), we get

$$\varrho(\zeta, \varepsilon) \leq \lambda \varrho(\zeta, \varepsilon),$$

as a result we have, $\varrho(\zeta, \varepsilon) = 0$, since $\lambda < 1$. Hence, $\zeta = \varepsilon$. In particular, $\zeta = \varepsilon = \sup\{\mathcal{J}^n \zeta\}$ in consequence, we get $\mathcal{J}\zeta \leq \zeta$, a contradiction. Therefore, $\mathcal{J}\zeta = \zeta$.

Case 2: If $\varrho(\mathcal{J}^n \zeta, \mathcal{J}\mathcal{U}_n) + \varrho(\mathcal{U}_n, \mathcal{J}^{n+1} \zeta) = 0$, then $\varrho(\mathcal{U}_{n+1}, \mathcal{J}^{n+1} \zeta) = 0$, which implies that, $\varrho(\zeta, \varepsilon) = 0$ as

for every $\lambda, \theta, \mu \in [0, 1)$ with $0 \leq \lambda + 2\theta + \mu < 1$ and, $(1, 0) \leq \mathcal{J}((1, 0)), (1, 0) \in \mathcal{Q}$. Thus, all assumptions of Theorem 7 are met. Hence, $(1, 0)$ and $(0, 1)$ are fixed points of \mathcal{J} . \square

Example 9 Let $\{(\mathcal{U}_n, \zeta_n)\} \subseteq \mathcal{Q}$ be a non-decreasing sequence which converges to (\mathcal{U}, ζ) in Example 8. Then $\lim_{n \rightarrow \infty} (\mathcal{U}_n, \zeta_n) = (\mathcal{U}, \zeta)$, where (\mathcal{U}, ζ) is an upper bound as well as supreme of all terms of the sequence. Therefore, all assumptions in Theorem 8 are met and, $(1, 0)$ and $(0, 1)$ are the fixed points of \mathcal{J} in \mathcal{Q} .

If \mathcal{Q} satisfies the below condition, then the unique fixed point for \mathcal{J} exists in Theorems 7 and 8.

$$\begin{aligned} &\text{for any } \varepsilon, \zeta \in \mathcal{Q}, \text{ there exists } \mathcal{U} \\ &\in \mathcal{Q} \text{ which is comparable to } \varepsilon \text{ and } \zeta. \end{aligned} \quad (11)$$

Theorem 9 In addition \mathcal{Q} satisfies the condition (11) in Theorems 7 and 8, then one can obtain the unique fixed point of \mathcal{J} .

Proof We discuss the proof in the following subsequent cases.

Case 1: If $\varepsilon \neq \zeta$ are comparable, then we distinguish the below cases again.

(i). If $\varrho(\zeta, \mathcal{J} \varepsilon) + \varrho(\varepsilon, \mathcal{J} \zeta) \neq 0$ then Eq. (8) follows that,

$$\begin{aligned} \varrho(\varepsilon, \zeta) &= \varrho(\mathcal{J} \varepsilon, \mathcal{J} \zeta) \\ &\leq \lambda \varrho(\varepsilon, \zeta) + \theta [\varrho(\varepsilon, \mathcal{J} \varepsilon) + \varrho(\zeta, \mathcal{J} \zeta)] + \mu \frac{\varrho(\varepsilon, \mathcal{J} \varepsilon) \varrho(\varepsilon, \mathcal{J} \zeta) + \varrho(\zeta, \mathcal{J} \varepsilon) \varrho(\zeta, \mathcal{J} \zeta)}{\varrho(\zeta, \mathcal{J} \varepsilon) + \varrho(\varepsilon, \mathcal{J} \zeta)} \\ &\leq \lambda d(\varepsilon, \zeta) + \theta [\varrho(\varepsilon, \varepsilon) + \varrho(\zeta, \zeta)] \\ &\quad + \mu \frac{\varrho(\varepsilon, \varepsilon) \varrho(\varepsilon, \zeta) + \varrho(\zeta, \varepsilon) \varrho(\zeta, \zeta)}{\varrho(\zeta, \varepsilon) + \varrho(\varepsilon, \zeta)} \\ &\leq \lambda \varrho(\varepsilon, \zeta), \end{aligned}$$

a contradiction as $\lambda < 1$. Therefore, $\varepsilon = \zeta$.

(ii). If $\varrho(\zeta, \mathcal{J} \varepsilon) + \varrho(\varepsilon, \mathcal{J} \zeta) = 0$, then $\varrho(\varepsilon, \zeta) = 0$, a contradiction to $\varepsilon \neq \zeta$. Hence, $\varepsilon = \zeta$.

Case 2: Suppose ε, ζ are not comparable, then from (11) there exists $\bar{U} \in \mathcal{Q}$ is comparable to ε, ζ . Besides, the monotone property suggest that $\mathcal{J}^n \bar{U}$ is comparable to $\mathcal{J}^n \varepsilon = \varepsilon$ and $\mathcal{J}^n \zeta = \zeta$ for $n \in \mathbb{N}$.

If $\mathcal{J}^{n_0} \bar{U} = \varepsilon$, for certain $n_0 \geq 1$, then $\{\mathcal{J}^n \bar{U} : n \geq n_0\}$ is a constant sequence, since ε is a fixed point. As a result, we get $\lim_{n \rightarrow \infty} \mathcal{J}^n \bar{U} = \varepsilon$. Assume, if $\mathcal{J}^n \bar{U} \neq \varepsilon$ for $n \geq 1$ then the following subcases, we have

(i). If $\varrho(\mathcal{J}^{n-1} \varepsilon, \mathcal{J}^n \bar{U}) + \varrho(\mathcal{J}^{n-1} \bar{U}, \mathcal{J}^n \varepsilon) \neq 0$, then for $n \geq 2$, (8) becomes,

$$\begin{aligned} \varrho(\mathcal{J}^n \bar{U}, \varepsilon) &= \varrho(\mathcal{J}^n \bar{U}, \mathcal{J}^n \varepsilon) \\ &\leq \lambda \varrho(\mathcal{J}^{n-1} \bar{U}, \varepsilon) \\ &\quad + \theta [\varrho(\mathcal{J}^{n-1} \bar{U}, \mathcal{J}^n \bar{U}) + \varrho(\varepsilon, \varepsilon)] \\ &\quad + \mu \frac{\varrho(\mathcal{J}^{n-1} \bar{U}, \mathcal{J}^n \bar{U}) \varrho(\mathcal{J}^{n-1} \bar{U}, \varepsilon) + \varrho(\varepsilon, \mathcal{J}^n \bar{U}) \varrho(\varepsilon, \varepsilon)}{\varrho(\mathcal{J}^n \bar{U}, \varepsilon) + \varrho(\varepsilon, \mathcal{J}^{n-1} \bar{U})} \\ &\leq \lambda \varrho(\mathcal{J}^{n-1} \bar{U}, \varepsilon) + \theta [\varrho(\mathcal{J}^{n-1} \bar{U}, \varepsilon) + \varrho(\varepsilon, \mathcal{J}^n \bar{U})] \\ &\quad + \mu \varrho(\mathcal{J}^{n-1} \bar{U}, \varepsilon). \end{aligned}$$

Thus,

$$\varrho(\mathcal{J}^n \bar{U}, \varepsilon) \leq \left(\frac{\lambda + \theta + \mu}{1 - \theta} \right) \varrho(\mathcal{J}^{n-1} \bar{U}, \varepsilon).$$

Inductively, we get

$$\varrho(\mathcal{J}^n \bar{U}, \varepsilon) \leq \left(\frac{\lambda + \theta + \mu}{1 - \theta} \right)^n \varrho(\bar{U}, \varepsilon), \quad (12)$$

which results in $\mathcal{J}^n \bar{U} \rightarrow \varepsilon$ as $n \rightarrow \infty$ in Eq. (12). As by the same argument, we obtain that $\mathcal{J}^n \bar{U} \rightarrow \zeta$ as $n \rightarrow \infty$. The result of uniqueness implies that, $\varepsilon = \zeta$.

(ii). If $\varrho(\mathcal{J}^{n-1} \varepsilon, \mathcal{J}^n \bar{U}) + \varrho(\mathcal{J}^{n-1} \bar{U}, \mathcal{J}^n \varepsilon) = 0$, then

$\varrho(\mathcal{J}^n \bar{U}, \varepsilon) = 0$. Therefore, $\lim_{n \rightarrow \infty} \mathcal{J}^n \bar{U} = \varepsilon$. Also from the similar argument, we acquire that, $\lim_{n \rightarrow \infty} \mathcal{J}^n \bar{U} = \zeta$. Hence, $\varepsilon = \zeta$. \square

Example 10 Let $C[0, 1] = \{\bar{U} : [0, 1] \rightarrow \mathbb{R}, \text{continuous}\}$ be a space with the partial order

$$\bar{U} \leq \zeta \text{ if and only if } \bar{U}(t) \leq \zeta(t), \text{ for } t \in [0, 1],$$

and let the metric be

$$\varrho(\bar{U}, \zeta) = \sup\{|\bar{U}(t) - \zeta(t)| : t \in [0, 1]\},$$

satisfies the condition (8) and $\max\{\bar{U}, \zeta\}(t) = \max\{\bar{U}(t), \zeta(t)\}$ is continuous. Furthermore, $(C[0, 1], \leq)$ satisfies the condition (11) and hence the uniqueness.

Example 11 Let $\mathcal{Q} = \{(0, 0), (\frac{1}{2}, 0), (0, 1)\}$ be a subset of \mathbb{R}^2 with the order \leq defined by: for $(\bar{U}_1, \zeta_1), (\bar{U}_2, \zeta_2) \in \mathcal{Q}$ with $(\bar{U}_1, \zeta_1) \leq (\bar{U}_2, \zeta_2)$ if and only if $\bar{U}_1 \leq \bar{U}_2$ and $\zeta_1 \leq \zeta_2$. A metric $\varrho : \mathcal{Q} \times \mathcal{Q} \rightarrow \mathbb{R}$ is defined by

$$\varrho((\bar{U}_1, \zeta_1), (\bar{U}_2, \zeta_2)) = \max\{|\bar{U}_1 - \bar{U}_2|, |\zeta_1 - \zeta_2|\}.$$

A self map \mathcal{J} on \mathcal{Q} is defined by $\mathcal{J}(0, 0) = (0, 0)$, $\mathcal{J}(0, 1) = (\frac{1}{2}, 0)$ and $\mathcal{J}(\frac{1}{2}, 0) = (0, 0)$. Therefore, all the assumptions of Theorems 7, 8 and 9 are met and hence, \mathcal{J} has a unique fixed point $(0, 0) \in \mathcal{Q}$.

Remark 1

- (a). Theorems 2.1, 2.2 and 2.3 of [20] can be found from Theorems 7, 8 and 9 by setting $\theta = \mu = 0$.
- (b). By replacing $\theta = 0$ in Theorems 7, 8 and 9, we obtain Theorems 15, 16 and 18 of [26].
- (c). Theorem 20 of [26] can get by putting $\lambda = \theta = 0$ in Theorem 7.

Some consequences from Section 0.1 can get by taking $\lambda = 0$ and $\theta = 0$.

Theorem 10 Suppose $(\mathcal{Q}, \varrho, \preceq)$ is a complete partially ordered metric space. A non-decreasing self map \mathcal{J} on \mathcal{Q} satisfies the below contraction condition for every $\mathcal{U} \neq \varepsilon \in \mathcal{Q}$ with $\varepsilon \preceq \mathcal{U}$ then \mathcal{J} has a fixed point, if $\mathcal{U}_0 \preceq \mathcal{J} \mathcal{U}_0$, for certain $\mathcal{U}_0 \in \mathcal{Q}$.

$$\varrho(\mathcal{J} \mathcal{U}, \mathcal{J} \varepsilon) \leq \begin{cases} \theta [\varrho(\mathcal{U}, \mathcal{J} \mathcal{U}) + \varrho(\varepsilon, \mathcal{J} \varepsilon)] + \mu \frac{\varrho(\mathcal{U}, \mathcal{J} \mathcal{U}) \varrho(\mathcal{U}, \mathcal{J} \varepsilon) + \varrho(\varepsilon, \mathcal{J} \mathcal{U}) \varrho(\varepsilon, \mathcal{J} \varepsilon)}{\varrho(\varepsilon, \mathcal{J} \mathcal{U}) + \varrho(\mathcal{U}, \mathcal{J} \varepsilon)}, & \text{if } A \neq 0 \\ 0, & \text{if } A = 0 \end{cases} \quad (13)$$

where $A = \varrho(\varepsilon, \mathcal{J} \mathcal{U}) + \varrho(\mathcal{U}, \mathcal{J} \varepsilon)$, and θ, μ are non-negative reals with $0 \leq 2\theta + \mu < 1$ and either \mathcal{J} is continuous or \mathcal{Q} has an ordered complete(OC) property.

Theorem 11 A non-decreasing self map \mathcal{J} on \mathcal{Q} , where $(\mathcal{Q}, \varrho, \preceq)$ be a complete partially ordered metric space has a fixed point, if it satisfies the below contraction condition for every $\mathcal{U} \neq \varepsilon \in \mathcal{Q}$ with $\varepsilon \preceq \mathcal{U}$ and $\mathcal{U}_0 \preceq \mathcal{J} \mathcal{U}_0$, for certain $\mathcal{U}_0 \in \mathcal{Q}$.

$$\varrho(\mathcal{J} \mathcal{U}, \mathcal{J} \varepsilon) \leq \begin{cases} \lambda \varrho(\mathcal{U}, \varepsilon) + \mu \frac{\varrho(\mathcal{U}, \mathcal{J} \mathcal{U}) \varrho(\mathcal{U}, \mathcal{J} \varepsilon) + \varrho(\varepsilon, \mathcal{J} \mathcal{U}) \varrho(\varepsilon, \mathcal{J} \varepsilon)}{\varrho(\varepsilon, \mathcal{J} \mathcal{U}) + \varrho(\mathcal{U}, \mathcal{J} \varepsilon)}, & \text{if } A \neq 0 \\ 0, & \text{if } A = 0 \end{cases} \quad (14)$$

where $A = \varrho(\varepsilon, \mathcal{J} \mathcal{U}) + \varrho(\mathcal{U}, \mathcal{J} \varepsilon)$ and λ, μ are non-negative reals with $0 \leq \lambda + \mu < 1$ and either \mathcal{J} is continuous or \mathcal{Q} has an ordered complete(OC) property.

Besides, a unique fixed point for \mathcal{J} can be obtained from Theorems 10 and 11, if \mathcal{Q} satisfies the condition (11).

Theorem 12 Suppose $(\mathcal{Q}, d, \preceq)$ is a complete partially ordered metric space. A nondecreasing self map \mathcal{J} on \mathcal{Q} satisfies the condition (1) (or (5)) for some $\mathcal{U}_0 \in \mathcal{Q}$ with $\mathcal{U}_0 \succeq \mathcal{J} \mathcal{U}_0$, and is either continuous or \mathcal{Q} satisfies

if a non-increasing sequence $\{\mathcal{U}_n\} \rightarrow \mathcal{U}$ in \mathcal{Q} , then $\mathcal{U} = \inf\{\mathcal{U}_n\}$.

Then \mathcal{J} has a fixed point in \mathcal{Q} .

Proof The procedure of the proof follows Theorems 7 and 8. \square

Theorem 13 Condition (11) gives the uniqueness of a fixed point of \mathcal{J} in Theorem 12.

Remark 2 In [20], instead of condition (11), the authors use the following weaker condition:

$$\begin{aligned} &\text{if a non-decreasing (non-increasing) sequence} \\ &\{\mathcal{U}_n\} \rightarrow \mathcal{U} \text{ in } \mathcal{Q}, \text{ then} \\ &\mathcal{U}_n \preceq \mathcal{U} (\mathcal{U} \preceq \mathcal{U}_n), \text{ for all } n \in \mathbb{N}. \end{aligned} \quad (15)$$

we have not been able to prove Theorem 1 and 8 and its consequences using (15).

We use the following definitions in the upcoming corollaries.

Definition 5 Let \mathcal{Q} be a nonempty set and $\mathcal{U}_0 \in \mathcal{Q}$. Let \mathcal{U}_0 . The orbit of \mathcal{U}_0 is defined by $\mathcal{O}(\mathcal{U}_0) = \{\mathcal{U}_0, \mathcal{J} \mathcal{U}_0, \mathcal{J}^2 \mathcal{U}_0, \dots\}$.

Definition 6 Let (\mathcal{Q}, ϱ) be a metric space and $\mathcal{J} : \mathcal{Q} \rightarrow \mathcal{Q}$. \mathcal{Q} is said to be \mathcal{J} -orbitally complete if every Cauchy sequence in $\mathcal{O}(\mathcal{U})$, $\mathcal{U} \in \mathcal{Q}$, converges to a point in \mathcal{Q} .

Definition 7 Let (\mathcal{Q}, ϱ) be a metric space and $\mathcal{J} : \mathcal{Q} \rightarrow \mathcal{Q}$. \mathcal{J} is said to be orbitally continuous at $\mu \in \mathcal{Q}$ if $\mathcal{J} \mathcal{U}_n \rightarrow \mathcal{J} \mu$ as $n \rightarrow \infty$ whenever $\mathcal{U}_n \rightarrow \mu$ as $n \rightarrow \infty$.

Now a consequence of the main result in terms integral type contractions for an orbitally complete partially ordered metric space is as follows.

Corollary 4 Let $(\mathcal{Q}, \varrho, \preceq)$ be a \mathcal{J} -orbitally complete partially ordered metric space. A non-decreasing self map \mathcal{J} on \mathcal{Q} satisfies,

$$\int_0^{\varrho(\mathcal{J}\mathcal{U}, \mathcal{J}\varepsilon)} d\Lambda \leq \alpha \int_0^{\frac{\varrho(\mathcal{U}, \mathcal{J}\mathcal{U})\varrho(\varepsilon, \mathcal{J}\varepsilon)}{\varrho(\mathcal{U}, \varepsilon)}} d\Lambda + \beta \int_0^{\varrho(\mathcal{U}, \mathcal{J}\mathcal{U}) + \varrho(\varepsilon, \mathcal{J}\varepsilon)} d\Lambda + \gamma \int_0^{\varrho(\mathcal{U}, \varepsilon)} d\Lambda + \mathcal{L} \int_0^{\min\{\varrho(\mathcal{U}, \mathcal{J}\varepsilon), \varrho(\varepsilon, \mathcal{J}\mathcal{U})\}} d\Lambda, \quad (16)$$

for every $\mathcal{U} \neq \varepsilon \in \mathcal{Q}$ with $\mathcal{U} \preceq \varepsilon$ and there exist $\alpha, \beta, \gamma \in [0, 1]$ with $0 < \alpha + 2\beta + \gamma < 1$, and $\mathcal{L} \geq 0$. If $\mathcal{U}_0 \preceq \mathcal{J}\mathcal{U}_0$, for certain $\mathcal{U}_0 \in \mathcal{Q}$, then \mathcal{J} has at least one fixed point in \mathcal{Q} .

Similarly, the following result is the consequence of Corollary 1.

Corollary 5 A non-decreasing continuous self-map \mathcal{J} on \mathcal{Q} satisfies the below condition for every $\mathcal{U} \neq \varepsilon \in \mathcal{Q}$ with $\mathcal{U} \preceq \varepsilon$, then \mathcal{J} has a fixed point, if $\mathcal{U}_0 \preceq \mathcal{J}\mathcal{U}_0$ for certain $\mathcal{U}_0 \in \mathcal{Q}$.

$$\int_0^{\varrho(\mathcal{J}\mathcal{U}, \mathcal{J}\varepsilon)} d\Lambda \leq \alpha \int_0^{\frac{\varrho(\mathcal{U}, \mathcal{J}\mathcal{U})\varrho(\varepsilon, \mathcal{J}\varepsilon)}{\varrho(\mathcal{U}, \varepsilon)}} d\Lambda + \beta \int_0^{\varrho(\mathcal{U}, \mathcal{J}\mathcal{U}) + \varrho(\varepsilon, \mathcal{J}\varepsilon)} d\Lambda + \gamma \int_0^{\varrho(\mathcal{U}, \varepsilon)} d\Lambda, \quad (17)$$

where $\alpha, \beta, \gamma \in [0, 1]$ with $0 < \alpha + 2\beta + \gamma < 1$.

Limitations

In complete partially ordered metric space, the existence of a fixed point of a self mapping satisfying generalized contraction of rational type is discussed. The uniqueness of a fixed point of the mapping is also obtained under an order relation in the space. Suitable examples are given at all possible stages to support the new findings. Some of these results are generalized and extended the well-known results in an ordered metric space. Also a result is widened from general metric space to partially ordered metric spaces with suitable example. A few consequences of the main results in terms of integral contractions are presented at the end.

- The results can be extended for a mapping in partially ordered b -metric space to acquire a fixed point.
- We can also obtain a coincidence point, common fixed point, coupled fixed point and coupled common fixed points by involving two mappings of the contraction conditions in partially ordered b -metric with required topological properties like monotone non-decreasing, mixed monotone, compatible etc.

Acknowledgements

The authors do thankful to the editor and anonymous reviewers for their valuable suggestions and comments which improved the contents of the paper.

Author's contributions

NSR contributed in the conceptualization, formal analysis, methodology, writing, editing and approving the manuscript. KK involved in formal analysis, methodology, writing and supervising the work. All authors read and approved the final manuscript.

Funding

Not applicable.

Availability of data and materials

Not applicable.

Declarations

Ethics approval and consent to participate

Not applicable.

Consent for publication

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author details

¹Department of Applied Mathematics, School of Applied Natural Sciences, Adama Science and Technology University, Post Box No. 1888, Adama, Ethiopia. ²Department of Mathematics, Vignan's Foundation for Science, Technology and Research, Vadlamudi 522213, Andhra Pradesh, India.

Received: 16 August 2021 Accepted: 29 September 2021

Published online: 15 October 2021

References

1. Banach S. Sur les operations dans les ensembles abstraits et leur application aux equations integrales. *Fund Math.* 1922;3:133–81.
2. Dass BK, Gupta S. An extension of Banach contraction principle through rational expression. *Indian J Pure Appl Math.* 1975;6:1455–8.
3. Chatterjee SK. Fixed point theorems. *C R Acad Bulgara Sci.* 1972;25:727–30.
4. Edelstein M. On fixed points and periodic points under contraction mappings. *J Lond Math Soc.* 1962;3:74–9.
5. Hardy GC, Rogers T. A generalization of fixed point theorem of S. Reich. *Can Math Bull.* 1973;16:201–6.
6. Jaggi DS. Some unique fixed point theorems. *Indian J Pure Appl Math.* 1977;8:223–30.
7. Kannan R. Some results on fixed points-II. *Am Math Mon.* 1969;76:71–6.
8. Reich S. Some remarks concerning contraction mappings. *Can Math Bull.* 1971;14:121–4.
9. Singh MR, Chatterjee AK. Fixed point theorems. *Commun Fac Sci Univ Ank Ser.* 1988;A1(37):1–4.
10. Smart DR. *Fixed Point Theorems*. Cambridge: Cambridge University Press; 1974.
11. Wong CS. Common fixed points of two mappings. *Pac J Math.* 1973;48:299–312.
12. Zamfirescu T. Fixed point theorems in metric spaces. *Arch Math.* 1972;23:292–8.
13. Agarwal RP, El-Gebeily MA, O'Regan D. Generalized contractions in partially ordered metric spaces. *Appl Anal.* 2008;87:1–8.
14. Altun I, Damjanovic B, Djoric D. Fixed point and common fixed point theorems on ordered cone metric spaces. *Appl Math Lett.* 2010;23:310–6.
15. Amini-Harandi A, Emami AH. A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations. *Nonl Anal Theo Methods Appl.* 2010;72:2238–42.

16. Arshad M, Azam A, Vetro P. Some common fixed results in cone metric spaces. *Fixed Point Theory Appl.* 2009. Article ID 493965
17. Arshad M, Ahmad J, Karapinar E. Some common fixed point results in rectangular metric space. *Int J Anal.* 2013. Article ID 307234
18. Bhaskar TG, Lakshmikantham V. Fixed point theory in partially ordered metric spaces and applications. *Nonl Anal Theo Methods Appl.* 2006;65:1379–93.
19. Hong S. Fixed points of multivalued operators in ordered metric spaces with applications. *Nonl Anal Theo Methods Appl.* 2010;72:3929–42.
20. Nieto JJ, Lopez RR. Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. *Order.* 2005;22:223–39.
21. Nieto JJ, Lopez RR. Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equation. *Acta Math Sin Engl Ser.* 2007;23(12):2205–12.
22. Ozturk M, Basarir M. On some common fixed point theorems with rational expressions on cone metric spaces over a Banach algebra. *Hacet J Math Stat.* 2012;41(2):211–22.
23. Ran ACM, Reurings MCB. A fixed point theorem in partially ordered sets and some application to matrix equations. *Proc Am Math Soc.* 2004;132:1435–43.
24. Wolk ES. Continuous convergence in partially ordered sets. *Gen Topol Appl.* 1975;5:221–34.
25. Zhang X. Fixed point theorems of multivalued monotone mappings in ordered metric spaces. *Appl Math Lett.* 2010;23:235–40.
26. Arshad M, Karapinar E, Ahmad J. Some unique fixed point theorems for rational contractions in partially ordered metric spaces. *J Inequal Appl.* 2013;248.
27. Azam A, Fisher B, Khan M. Common fixed point theorems in complex valued metric spaces. *Numer Funct Anal Optim.* 2011;32(3):243–53.
28. Karapinar E. Couple fixed point on cone metric spaces. *Gazi Univ J Sci.* 2011;24(1):51–8.
29. Aydi H, Karapinar E, Shatanawi W. Coupled fixed point results for (ψ, φ) -weakly contractive condition in ordered partial metric spaces. *Comput Math Appl.* 2011;62(12):4449–60.
30. Choudhury BS, Kundu A. A coupled coincidence point result in partially ordered metric spaces for compatible mappings. *Nonl Anal Theo Methods Appl.* 2010;73:2524–31.
31. Ćirić LB, Olatinwo MO, Gopal D, Akinbo G. Coupled point theorems for mappings satisfying a contractive condition of rational type on a partially ordered metric space. *Adv Fixed Point Theo.* 2012;2:1–8.
32. Lakshmikantham V, Ćirić LB. Couple fixed point theorems for nonlinear contractions in partially ordered metric spaces. *Nonl Anal Theo Methods Appl.* 2009;70:4341–9.
33. Luong NV, Thuan NX. Coupled fixed points in partially ordered metric spaces and application. *Nonl Anal Theo Methods Appl.* 2011;74:983–92.
34. Seshagiri Rao N, Kalyani K. Coupled fixed point theorems with rational expressions in partially ordered metric spaces. *J Anal.* 2020;28(4):1085–95. <https://doi.org/10.1007/s41478-020-00236-y>.
35. Seshagiri Rao N, Kalyani K, Kejal K. Contractive mapping theorems in Partially ordered metric spaces. *CUBO.* 2020;22(2):203–14.
36. Seshagiri Rao N, Kalyani K. Unique fixed point theorems in partially ordered metric spaces. *Heliyon.* 2020;6(11):e05563. <https://doi.org/10.1016/j.heliyon.2020.e05563>.
37. Kalyani K, Seshagiri Rao N. Coincidence point results of nonlinear contractive mappings in partially ordered metric spaces. *CUBO.* 2021;23(2):207–24.
38. Kalyani K, Seshagiri Rao N, Belay M. On fixed point theorems of monotone functions in Ordered metric spaces. *J Anal.* 2021;14. <https://doi.org/10.1007/s41478-021-00308-7>.
39. Seshagiri Rao N, Kalyani K. On some coupled fixed point theorems with rational expressions in partially ordered metric spaces. *Sahand Commun Math Anal (SCMA).* 2021;18(1):123–36. <https://doi.org/10.22130/scma.2020.120323.739>.
40. Seshagiri Rao N, Kalyani K. Generalized contractions to coupled fixed point. Theorems in partially ordered metric spaces. *J Siberian Fed Univ Math Phys.* 2020;23(4):492–502. <https://doi.org/10.17516/1997-1397-2020-13-4-492-502>.
41. Seshagiri Rao N, Kalyani K. Coupled fixed point theorems in partially ordered metric spaces. *Fasciculi Mathematic.* 2020;64:77–89. <https://doi.org/10.21008/fj.0044-4413.2020.0011>
42. Kalyani K, Seshagiri Rao N, Belay Mitiku. Some fixed point results in ordered b -metric space with auxiliary function. *Advances in the Theory of Nonlinear Analysis and its Application (ATNAA).* 2021;5(3):421–432. <https://doi.org/10.31197/atnaa.758962>
43. Seshagiri Rao N, Kalyani K, Belay Mitiku. Fixed point theorems for nonlinear contractive mappings in ordered b -metric space with auxiliary function. *BMC Res Notes.* 2020;13:451. <https://doi.org/10.1186/s13104-020-05273-1>.
44. Belay Mitiku, Seshagiri Rao N, Kalyani K. Some fixed point results of generalized (ϕ, ψ) -contractive mappings in ordered b -metric Spaces. *BMC Res Notes.* 2020;13:537. <https://doi.org/10.1186/s13104-020-05354-1>.
45. Kalyani K, Seshagiri Rao N, Belay Mitiku. Fixed point results of contractive mappings with altering distance functions in ordered b -metric spaces. *Informat Sci Lett.* 2021;10(2):267–75. <https://doi.org/10.18576/isl/100211>.
46. Seshagiri Rao N, Kalyani K. Some fixed point results of (ϕ, ψ, θ) -contractive mappings in ordered b -metric spaces. *Math Sci.* 2021;13. <https://doi.org/10.1007/s40096-021-00408-2>.
47. Seshagiri Rao N, Kalyani K, Prasad K. Fixed point results for weak contractions in partially ordered b -metric space. *BMC Res Notes.* 2021;14:263. <https://doi.org/10.1186/s13104-021-05649-x>.
48. Kalyani K, Seshagiri Rao N, Mishra LN. Coupled fixed points theorems for generalized weak contractions in ordered b -metric spaces. *Asian-Eur J Math.* 2022;22. <https://doi.org/10.1142/S1793557122500504>.
49. Acar Ö. Fixed point theorems for rational type F -contraction. *Carpathian Math Publ.* 2021;13(1):39–47. <https://doi.org/10.15330/cmp.13.1.39-47>.
50. Ameer Eskandar, Aydi Hassen, Arshad Muhammad, De la Sen Manuel. Hybrid Ćirić type graphic (Y, Δ) -contraction mappings with applications to electric circuit and fractional differential equations. *Symmetry (Basel).* 2020;12(2):467. <https://doi.org/10.3390/sym12030467>.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Ready to submit your research? Choose BMC and benefit from:

- fast, convenient online submission
- thorough peer review by experienced researchers in your field
- rapid publication on acceptance
- support for research data, including large and complex data types
- gold Open Access which fosters wider collaboration and increased citations
- maximum visibility for your research: over 100M website views per year

At BMC, research is always in progress.

Learn more biomedcentral.com/submissions

